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Forum for CSIR-UGC JRF/NET, GATE, IIT-JAM/IISc, JEST, TIFR and GRE in PHYSICS & PHYSICAL SCIENCES

Mechanics and General Properties of Matter

(IIT-JAM/JEST/TIFR/M.Sc Entrance)
MECHANICS

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1. Newton's Laws of Motion

1.1 Newton's laws of motion are three physical laws that, together, laid the foundation for classical mechanics. They describe the relationship between a body and the forces acting upon it, and its motion in response to those forces.

1.1.1 First Law: When viewed in an inertial reference frame, an object either remains at rest or continues to move at a constant velocity, unless acted upon by an external force.

The first law states that if the net force (the vector sum of all forces acting on an object) is zero, then the velocity of the object is constant. Velocity is a vector quantity which expresses both the object's speed and the direction of its motion; therefore, the statement that the object's velocity is constant is a statement that both its speed and the direction of its motion are constant. The first law can be stated mathematically as $\sum F = 0 \Rightarrow \frac{dv}{dt} = 0$.

Consequently,

- An object that is at rest will stay at rest unless an external force acts upon it.
- An object that is in motion will not change its velocity unless an external force acts upon it.

Newton placed the first law of motion to establish frames of reference for which the other laws are applicable. The first law of motion postulates the existence of at least one frame of reference called a Newtonian or inertial reference frame, relative to which the motion of a particle not subject to forces is a straight line at a constant speed. Newton's first law is often referred to as the law of inertia. Thus, a condition necessary for the uniform motion of a particle relative to an inertial reference frame is that the total net force acting on it is zero.

1.1.2 Second Law: The vector sum of the external forces $\vec{F}$ on an object is equal to the mass $m$ of that object multiplied by the acceleration vector $\vec{a}$ of the object $\vec{F} = m\vec{a}$. The second law states that the net force on an object is equal to the rate of change (that is, the derivative) of its linear momentum $\vec{p}$ in an inertial reference frame

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d(m\vec{v})}{dt}$$

The second law can also be stated in terms of an object's acceleration. Since Newton's second law is only valid for constant-mass systems, mass can be taken outside the differentiation operator by the constant factor rule in differentiation. Thus,
\[ \vec{F} = m \frac{d\vec{v}}{dt} = m\vec{a} \]

where \( \vec{F} \) is the net force applied, \( m \) is the mass of the body, and \( \vec{a} \) is the body's acceleration. Thus, the net force applied to a body produces a proportional acceleration. In other words, if a body is accelerating, then there is a force on it.

Consistent with the first law, the time derivative of the momentum is non-zero when the momentum changes direction, even if there is no change in its magnitude; such is the case with uniform circular motion. The relationship also implies the conservation of momentum: when the net force on the body is zero, the momentum of the body is constant. Any net force is equal to the rate of change of the momentum.

Any mass that is gained or lost by the system will cause a change in momentum that is not the result of an external force. A different equation is necessary for variable-mass systems. Newton's second law requires modification if the effects of special relativity are to be taken into account, because at high speeds the approximation that momentum is the product of rest mass and velocity is not accurate.

**1.1.3 Third Law:** When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

An impulse \( \vec{J} \) occurs when a force \( \vec{F} \) acts over an interval of time \( \Delta t \), and it is given by

\[ \vec{J} = \int \vec{F} \, dt \]

Since force is the time derivative of momentum, it follows that

\[ \vec{J} = \Delta \vec{p} = m \Delta \vec{v} \]

This relation between impulse and momentum is closer to Newton's wording of the second law.

**1.2 Application of Newton's Law of Motion**

To apply Newton's law of motion one should follow following step.

Step 1- Draw free body diagram and identified external forces

Step 2 – Write down equation of constrain.

Step 3 – Write down Newton’s law of motion.
1.2.1 Free Body Diagram

To draw free body diagram assume body is point mass of mass \( m \) then assume coordinate axis as \( x \)-axis is direction of motion , the perpendicular to plane of motion is \( y \)-axis , after determining the axis find all type of external forces.

Some of common external forces are discuss below.

1. **Weight**: The weight of body is \( w = mg \). The weight of the body is always in vertically downward direction, as shown in figure.

   ![Free Body Diagram](image)

   2. **Normal forces** when body of mass touching any surface then surface will exert normal force \( N \). The direction of normal force is perpendicular to plane of surface.

   ![Normal Forces Diagram](image)

   3. **Tension**: Force \( F \) is exerted on mass \( m \) through string. Then any section of string is pulled by two equal and opposite forces. Any one of these forces is called tension. Tension always gives pulling effect. In the figure, the force \( F \) is acted on mass \( m \) through string so there is tension \( T \) in the string giving pulling effect.

      ![Tension Diagram](image)

Now let us analyze another system in which two masses of mass \( m_1 \) and \( m_2 \) attached with string and force \( F \) acted on mass \( m_1 \) as shown in fig. Due to force \( F \) tension \( T \) developed in string if one will draw free body diagram for \( m_1 \) the tension will act towards left and if free body diagram draw for \( m_2 \) the tension will act towards right. In both the case tension will produce pulling effect on mass \( m_1 \) and \( m_2 \).

   ![Tension in System Diagram](image)

   - Same thread same tension
   - Different thread different tension
If the rope is mass less the tension in string is same for all sections of string otherwise it will depend on mass of segment of the rope.

4. Friction: in general friction force is force which is responsible to oppose the motion.

There is two type of frictional force

a) **Static Friction**: when there is not any relative motion between surface and body \( m \) then frictional identified as static frictional which is force is equal to external force acted on mass \( m \). the direction of frictional force is tangent or parallel to surface. if \( \mu \) is coefficient of friction between surface and mass \( m \) then maximum value of frictional force is \( \mu N \).

In summary we take the force of friction \( f \) to behave as follows

\[ 0 \leq f \leq \mu N \] where \( f \) oppose the motion that would occur in its absence

b) **Kinetic Friction**: if there is relative speed between surface and mass \( m \) then frictional force is identifying as \( f_k = \mu N \) where \( \mu \) is coefficient of kinetic friction and \( N \) is normal force. the direction of \( f \) is directed opposite to the motion.

**Example**: Block and Wedge with Friction

A block of mass \( m \) rests on a fixed wedge of angle \( \theta \). The coefficient of friction is \( \mu \). (For wooden blocks, \( \mu \) is of the order of 0.2 to 0.5). Find the value of \( \theta \) at which the block starts to slide.

**Solution**: In the absence of friction. The block would slide down the plane; hence the friction force \( f \) points up the plane. With the coordinate shown, we have

\[ m \dot{x} = W \sin \theta - f \quad \text{and} \quad m \dot{y} = N - W \cos \theta = 0 \]

when sliding starts. \( f \) has its maximum value \( \mu N \), and \( \ddot{x} = 0 \). The equation then give

\[ W \sin \theta_{\max} = \mu N \]

\[ W \cos \theta_{\max} = N \]

Hence, \( \tan \theta_{\max} = \mu \)

Notice that as the wedge angle is gradually increased from zero, the friction force grows in magnitude from zero toward its maximum value \( \mu N \) since before the block begins to slide we have \( f = W \sin \theta \quad \theta \leq \theta_{\max} \)
5. **Pseudo Force**: some time the observer is on accelerated frame of reference then observer will measured acceleration without any external force, which invalidate newton's law of motion. So one can use a concept of pseudo force. if $a_0$ is acceleration of observer and he is measuring the pseudo force $F_s$ on rest mass $m$. The magnitude of pseudo $F_s = ma_o$ and direction is opposite to direction of observer.

One can understand the concept of pseudo force from following example

Assume two observer $A$ and $B$ moving with acceleration $a_A\hat{i}$ and $-a_B\hat{i}$ respectively with respect to mass $m$ then observer $A$ measure pseudo force $ma_A$ is $\hat{i}$ direction and observer $B$ measure pseudo force $ma_B$ is $-\hat{i}$ direction respectively.

Concept of pseudo force makes Newton’s laws & motion valid in non-inertial frames.

1.2.2 **Equation of Constrain**: limit to motion is known as constrain motion. The relationship between variable which. Actually equation of constraint is used to find independent variable which will describe independent motion of the system. Constraint equations are independent of applied force. It is very important tool to find relationship between acceleration between different particles. which can be better understand by following examples

**Example**: A block moves on the wedge which in turns moves on a horizontal table. as shown in sketch, if the wedge angle is $\theta$. How the acceleration of wedge and block are related?

**Solution**: Let the $X$ be the horizontal coordinate of the end of the wedge and $x$ and $y$ be the horizontal and vertical coordinate of the block as shown in figure. Let $h$ is height of the wedge. So from geometry $(x-X)=(h-y)\cot\theta$ differentiation with respect to $t$

$$\ddot{x} - \ddot{X} = -\ddot{y}\cot\theta$$
Example: Two masses are connected by a string which passes over a pulley accelerating upward at rate \( a \) as shown in figure. If acceleration of the body 1 is \( a_1 \) and body 2 is \( a_2 \), what is relation between \( a, a_1 \) and \( a_2 \)?

Solution: We know that length of string is fixed so from the figure, \( l = \pi R + (y_p - y_1) + (y_p - y_2) \) differentiation with respect to \( t \)
\[
0 = 2\ddot{y}_p - \ddot{y}_1 - \ddot{y}_2
\]
\[
\ddot{y}_p = a, \quad \ddot{y}_1 = a_1, \quad \ddot{y}_2 = a_2 \quad \text{then} \quad a = \frac{a_1 + a_2}{2}.
\]

Example: the pulley system shown is fig how the acceleration of the rope compare the acceleration of the block. Let us assume \( x \) acceleration of the block is \( a \) and acceleration of end of rope is \( a_r \).

Again the length of rope is fixed let say \( l \). From the fig
\[
l = X + \pi R + (X - h) + \pi R + (x - h) .
\]

Where \( R \) is radius of pulley .differentiating with respect to
\[
0 = 2\ddot{X} + \ddot{x} \Rightarrow \ddot{X} = -\frac{\ddot{x}}{2} \quad \text{if} \quad \ddot{X} = a, \ddot{x} = a, \quad \text{then} \quad a = -\frac{a_r}{2}.
\]
1.2.3 Equation of Motion

Finally one should write equation of motion as \( \sum F_x = ma_x, \sum F_y = ma_y \) where \( \sum F_x \) is summation of external force is external force in \( x \) direction and \( \sum F_y \) is summation of external force is external force in \( y \) direction. \( a_x \) and \( a_y \) is acceleration in \( x \) and \( y \) direction.

Number of equation must be equal to number of unknown variable. so that one can know complete solution.

Example: Force \( F = 20N \) act on block \( m_1 = 5kg \) such that \( m_2 = 3kg \) is in contact with \( m_1 \) and \( m_3 = 2kg \) as shown in figure.

(i) Find acceleration of each block
(ii) Find contact force in each block

Solution: All are moving with same acceleration

\[ F = (m_1 + m_2 + m_3) a \Rightarrow a = 2 \text{ m/sec}^2 \]

Free body diagram
Example: The following parameters of the arrangement of are available: the angle $\alpha$ which the inclined plane forms with the horizontal, and the coefficient of friction $k$ between the body $m_1$ and the inclined plane. The masses of the pulley and the threads, as well as the friction in the pulley, are negligible. Assuming both bodies to be motionless at the initial moment, find the mass ratio $m_2/m_1$ at which the body $m_2$

(a) starts coming down;
(b) starts going up;
(c) is at rest.

Solution: (a) for $m_2$ starts coming down. For mass $m_2$, $m_2g > T$ and for mass $m_1$ moving up

$$T > m_1g \sin \alpha + f_{\text{max}}$$

therefore $m_2g > m_1g \sin \alpha + k m_1g \cos \alpha$

$$\frac{m_2}{m_1} > \sin \alpha + k \cos \alpha$$

for $m_2$ starts coming up. For mass $m_2$, $m_2g < T$ and for mass $m_1$ moving up

$$T < m_1g \sin \alpha - f_{\text{max}}$$

(b)

$$m_1g \sin \alpha > m_2g + km_1g \cos \alpha$$

$$g \sin \alpha > \frac{m_2}{m_1} + k \cos \alpha$$

$$\frac{m_2}{m_1} < g \sin \alpha - k \cos \alpha$$

(c) At rest: friction will be static:

$$g \sin \alpha - k \cos \alpha < \frac{m_2}{m_1} < \sin \alpha + k \cos \alpha$$
**Example:** At the moment $t = 0$ the force $F = at$ is applied to a small body of mass $m$ resting on smooth horizontal plane ($a$ is a constant). The permanent direction of this force forms an angle $\alpha$ with the horizontal. Find:

(a) the velocity of the body at the moment of its breaking off the plane;

(b) the distance traversed by the body up to this moment.

**Solution:**

(a) If one will draw the free body diagram $F \sin \alpha + N = mg$ in $y$ direction

$$F \cos \alpha = ma_1 \text{ in } x \text{ direction}.$$ 

At time of breaking off the plane vertical component of $F$ must be equal to weight $mg$. Then

$$F \sin \alpha = mg = at \sin \alpha$$

$$t = \frac{mg}{a \sin \alpha}$$

Motion equation of block: $a_1 = \text{Acceleration of block}$

$$F \cos \alpha = ma_1$$

$$a_1 = \frac{at \cos \alpha}{m} = \frac{dv}{dt}$$

$$\int_0^v \frac{mdv}{a \cos \alpha} = \int_0^{t_1} \frac{dt}{a \cos \alpha} \Rightarrow \frac{mv}{a \cos \alpha} = \frac{t_1^2}{2}$$

$$\frac{mv}{a \cos \alpha} = \frac{1}{2} \frac{m^2 g^2}{a^2 \sin^2 \alpha} \Rightarrow v = \frac{mg^2 \cos \alpha}{2a \sin^2 \alpha}$$

(b) $$\int_0^v \frac{mdv}{a \cos \alpha} = \int_0^{t_1} \frac{dt}{a \cos \alpha}$$

$$\frac{mv}{a \cos \alpha} = \frac{t^2}{2} \Rightarrow v = \frac{a \cos \alpha \cdot t^2}{2m} \Rightarrow \frac{ds}{dt} = \frac{a \cos \alpha \cdot t^2}{2m}$$

$$\int_0^t \frac{ds}{2m} = \frac{a \cos \alpha \cdot t^2}{2m} \Rightarrow s = \frac{m^2 g^3 \cos \alpha}{6a^2 \sin^3 \alpha}$$
Example: In the arrangement shown in fig the bodies have masses $m_0, m_1, m_2$, the friction is absent, the masses of the pulleys and the threads are negligible. Find the acceleration of the body $m_1$. Look into possible cases.

Solution: $T = 2T_1$ and from equation of constrain $a_0 = \frac{a_1 + a_2}{2}$

Equation of motion:

$$T = m_0 \left( \frac{a_1 + a_2}{2} \right) \quad \text{...(i)}$$

$$m_1 g - T / 2 = m_1 a_1 \quad \text{...(ii)}$$

$$m_2 g - T / 2 = m_2 a_2 \quad \text{...(iii)}$$

from (i), (ii) and (iii)

$$a_1 = \frac{4m_1 m_2 + m_0 (m_1 - m_2) g}{4 m_1 m_2 + m_0 (m_1 + m_2)}$$

Example: Coefficient of friction is $\mu$. What will be acceleration of table such that system will be in equilibrium?

Solution: This whole wedge is moving with acceleration $a_0$.

$$N = mg$$

$$ma_0 + fr = T \quad \Rightarrow ma_0 + \mu N = T$$

$$ma_0 + \mu mg = T \quad \Rightarrow m(a_0 + \mu g) = T \quad \text{(1)}$$

$$N - ma_0 = 0 \quad \Rightarrow N = ma_0$$

$$fr + T = mg \quad \Rightarrow T = mg - fr, \quad -\mu ma_0 + mg = T$$

$$\Rightarrow m(-\mu a_0 + g) = T \quad \text{(2)}$$

From (1) and (2) $ma_0 + \mu mg - mg + \mu ma_0 = 0$

$$-g(1-\mu) + a_0 (\mu + 1) = 0 \Rightarrow a_0 = g \frac{(1-\mu)}{(1+\mu)}$$
1.3 The Motion in Two Dimensional in Polar Coordinate

The drawing shows the unit vectors \( \hat{i}, \hat{j} \) and \( \hat{r}, \hat{\theta} \) at a point in the \( xy \) plane. We see that the orthogonality of \( \hat{r} \) and \( \hat{\theta} \) plus the fact that they are unit vectors,

\[
|\hat{r}| = 1, |\hat{\theta}| = 1,
\]

\( \hat{r} = \hat{i} \cos \theta + \hat{j} \sin \theta \) and \( \hat{\theta} = -\hat{i} \sin \theta + \hat{j} \cos \theta \) which is shown

The transformation can be shown by rotational Matrix

\[
\begin{bmatrix}
\dot{\hat{r}} \\
\dot{\hat{\theta}}
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\hat{i} \\
\hat{j}
\end{bmatrix}
\]

\[
\frac{d\hat{r}}{dt} = -\hat{i} \sin \theta \dot{\theta} + \hat{j} \cos \theta \dot{\theta} \Rightarrow \frac{d\hat{r}}{dt} = \dot{\theta} \hat{\theta}
\]

\[
\frac{d\hat{\theta}}{dt} = -\hat{i} \cos \theta \dot{\theta} - \hat{j} \sin \theta \dot{\theta} \Rightarrow \frac{d\hat{\theta}}{dt} = -\hat{r} \dot{\theta}
\]

1.3.1 The Position Vector in Polar Coordinate:

\[
\vec{r} = x\hat{i} + y\hat{j} \quad \vec{r} = r\left[\cos \theta \hat{i} + \sin \theta \hat{j}\right]
\]

\[
\hat{r} = \left|\vec{r}\right| \cos \theta \hat{i} + \left|\vec{r}\right| \sin \theta \hat{j}
\]

\[
\vec{r} = \left|\vec{r}\right| \left[\cos \theta \hat{i} + \sin \theta \hat{j}\right] \Rightarrow \vec{r} = \left|\vec{r}\right| \hat{r},
\]

\( \vec{r} = r \hat{r} \) is sometimes confusing, because the equation as written seems to make no reference to the angle \( \theta \). We know that two parameters needed to specify a position in two dimensional space (in Cartesian coordinates they are \( x \) and \( y \)), but the equation \( \vec{r} = r \hat{r} \) seems to contain only the quantity \( r \). The answer is that \( \hat{r} \) is not a fixed vector and we need to know the value of \( \theta \) to tell how \( \hat{r} \) is origin. Although \( \theta \) does not occur explicitly in \( r \hat{r} \), its value must be known to fix the direction of \( \hat{r} \). This would be apparent if we wrote \( \vec{r} = r \hat{r}(\theta) \) to emphasize the dependence of \( \hat{r} \) on \( \theta \). However, by common conversation \( \hat{r}(\theta) \) is understood to stand for \( \hat{r}(\theta) \).
1.3.2 Velocity Vector in Polar Coordinate:

\[ \vec{v} = \frac{d}{dt} (r\hat{r}) \Rightarrow \vec{v} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = \hat{r} + r\frac{d\hat{r}}{dt} \Rightarrow \vec{v} = \hat{r} + r\frac{d\hat{r}}{dt} \]

where \( \hat{r} \) is radial velocity in \( \hat{r} \) direction and \( r\frac{d\hat{r}}{dt} \) is tangential velocity in \( \hat{\theta} \) direction as shown in figure and the magnitude to velocity vector \( |\vec{v}| = \sqrt{\hat{r}^2 + r^2\hat{\theta}^2} \)

1.3.3 Acceleration Vector in Polar Coordinate

\[ \frac{d\vec{v}}{dt} = \frac{d\hat{r}}{dt}\hat{r} + \hat{r} \frac{d\hat{r}}{dt} + \frac{dr}{dt}\hat{\theta} + r\frac{d\hat{\theta}}{dt} + r\hat{\theta}\frac{d\hat{\theta}}{dt} \]

\[ \frac{d\vec{v}}{dt} = \hat{r} + r\frac{d\hat{r}}{dt} + \hat{r} \frac{d\hat{\theta}}{dt} + r\hat{\theta}\frac{d\hat{\theta}}{dt} \]

\[ \vec{a} = (\hat{r} - r\hat{\theta}^2)\hat{r} + (r\hat{\theta} + 2r\hat{\theta})\hat{\theta} \Rightarrow \vec{a} = a_r\hat{r} + a_\theta\hat{\theta} \]

\( a_r = \ddot{r} - r\dot{\theta}^2 \) is radial acceleration and \( a_\theta = r\ddot{\theta} + 2r\dot{\theta} \) is tangential acceleration.

So Newton’s law in polar coordinate can be written as

\[ f_r = ma_r = m(\ddot{r} - r\dot{\theta}^2) \] where \( f_r \) is force in radial direction.

\[ f_\theta = ma_\theta = m(r\ddot{\theta} + 2r\dot{\theta}) \] where \( f_\theta \) is force in tangential direction.

1.3.4 Circular Motion

For circular motion \( r = r_0 \) then \( \dot{r} = 0 \) so \( f_r = ma_r = -mr_0\dot{\theta}^2 \) where \( f_r \) is force in radial direction and \( f_\theta = ma_\theta = mr_0\dot{\theta} \) where \( f_\theta \) is force in tangential direction.

If there is not any force in tangential direction \( f_\theta = 0 \) is condition then motion is uniform circular motion ie \( \dot{\theta} = \omega \) is constant known as angular speed and tangential speed is given by \( v = r_0\omega \)
For non uniform circular motion radial acceleration is \( a_r = -\frac{v^2}{r} \) and tangential acceleration \( a_t = \frac{dv}{dt} \) is given by \( a_t = \frac{dv}{dt} \).

\[
a = \sqrt{a_r^2 + a_t^2}
\]

**Example:** The Spinning Terror

The Spinning Terror is an amusement park ride – a large vertical drum which spins so fast that everyone inside stays pinned against the wall when the floor drops away. What is the minimum steady angular velocity \( \omega \) which allows the floor to be dropped away safely?

**Solution:** Suppose that the radius of the drum is \( R \) and the mass of the body is \( M \). Let \( \mu \) be the coefficient of friction between the drum and \( M \). The forces on \( M \) are the weight \( W \), the friction force \( f \) and the normal force exerted by the wall, \( N \) as shown below.

The radial acceleration is \( R\omega^2 \) toward the axis, and the radial equation of motion is

\[
N = M / R\omega^2
\]

By the law of static friction,

\[
f \leq \mu N = \mu MR\omega^2
\]

Since we require \( M \) to be in vertical equilibrium,

\[
f = Mg,
\]

and we have

\[
Mg \leq \mu MR\omega^2
\]

or

\[
\omega^2 \geq \frac{g}{\mu R}
\]

The smallest value of \( \omega \) that will work is

\[
\omega_{\text{min}} = \sqrt{\frac{g}{\mu R}}
\]
Example: Mass \( m \) is whirled on the end of string \( R \) the motion is in a vertical plane in the gravitational field of earth. Write down the equation of motion in polar coordinate.

Solution: There are only two forces acted on body one tension \( T \) in radial direction and weight \( mg \) in vertical direction. Taking component of these two forces in \( \hat{r} \) and \( \hat{\theta} \) direction.

The Newton’s law of motion in radial direction is

\[
F_r = ma_r \text{ in } \hat{r} \text{ direction} \Rightarrow -T - mg \sin \theta = m(\ddot{r} - R \dot{\theta}^2)
\]

The Newton’s law of motion in tangential direction is

\[
F_\theta = ma_\theta \Rightarrow -mg \cos \theta = m(r \ddot{\theta} + 2\dot{r} \dot{\theta})
\]

The equation of constrain is \( r = R \quad \dot{r} = 0 \quad \ddot{r} = 0 \)

\[
-T - mg \sin \theta = -mR \ddot{\theta} \text{ in } \hat{r} \text{ direction}
\]

\[
-mg \cos \theta = mR \ddot{\theta} \Rightarrow \frac{d^2 \theta}{dt^2} = -\frac{g}{R} \cos \theta \text{ in } \hat{\theta} \text{ direction}
\]

Example: A horizontal frictionless table has a small hole in the centre of table. Block \( A \) of mass \( m_a \) on the table is connected by a block \( B \) of mass \( m_b \) hanging beneath by a string of negligible mass can move under gravity only in vertical direction. Which passes through the whole as shown in figure.

(a) Write down equation of motion in radial, tangential and vertical direction.

(b) What is equation of constrain?

(c) What will be the acceleration of \( B \) when \( A \) is moving with angular velocity \( \omega \) at radius \( r_0 \)?
Solution: Suppose of block $A$ rotating in circle with angular velocity $\omega$ of radius $r_0$ what is acceleration of block $B$ assume $\hat{z}$ direction is shown in figure.

The external force in radial direction is tension, not any external force in tangential direction and weight of body $m_b$ and tension in vertical direction.

(a) $m_a(\ddot{r} - r\dot{\theta}^2) = -T$ in $\hat{r}$ direction ..........(1)

$b_a(\dot{\theta} + 2\dot{r}\dot{\theta}) = 0$ in $\hat{\theta}$ direction ..........(2)

$m_b\ddot{z} = m_bg - T$ in $\hat{z}$ ..........(3)

(b) Since length of the rope is constant then

$r + z = l \Rightarrow \ddot{r} = -\ddot{z}$ ..........(4)

(c) Put value of $\ddot{r} = -\ddot{z}$ , $T = m_a(\ddot{z} + r\dot{\theta}^2)$

$\ddot{z} = \frac{m_bg - m_ar\dot{\theta}^2}{m_b + m_a}$ put $r = r_0$, $\dot{\theta} = \omega \Rightarrow \ddot{z} = \frac{m_bg - m_ar_0^2\omega^2}{m_a + m_b}$
MCQ (Multiple Choice Questions)

Q1. Two blocks $A$ and $B$ of masses $2m$ and $m$ respectively are connected by a massless and inextensible string. The whole system is suspended by a massless spring. The magnitudes of acceleration of $A$ and $B$, immediately after the string is cut, are respectively

(a) $\frac{g}{2}$, $g$
(b) $\frac{g}{2}$, $g$
(c) $g$, $g$
(d) $\frac{g}{2}$, $\frac{g}{2}$

Q2. A string of negligible mass going over a clamped pulley of mass $m$ supports a block of mass $M$ as shown in the figure. The force on the pulley by the clamp is given by

(a) $\sqrt{2} Mg$
(b) $\sqrt{2} mg$
(c) $\sqrt{(M+m)^2 + m^2} g$
(d) $\sqrt{(M+m)^2 + M^2} g$

Q3. What is the maximum value of the force $F$ such that the block shown in the arrangement, does not move? Take $g = 10 \text{ m/s}^2$.

(a) 20 N
(b) 10 N
(c) 12 N
(d) 15 N

Q4. A block $P$ of mass $m$ is placed on a horizontal frictionless plane. A second block of same mass $m$ is placed on it and is connected to a spring of constant $k$. The two blocks are pulled by distance $A$. Block $Q$ oscillates without slipping. What is the maximum value of frictional force between the two blocks?

(a) $\frac{kA}{2}$
(b) $kA$
(c) $\mu_s mg$
(d) zero
Q5. When forces $F_1, F_2$ and $F_3$ are acting on a particle of mass $m$ such that $F_2$ and $F_3$ are mutually perpendicular, then the particle remains stationary. If the force $F_1$ is now removed, then the acceleration of the particle

(a) $\frac{F_1}{m}$ (b) $\frac{F_2 F_3}{mF}$ (c) $\frac{F_2 - F_1}{m}$ (d) $\frac{F_2}{m}$

Q6. A smooth block is released at rest on a $45^\circ$ incline and then slides a distance $d$. If the time taken to slide on rough incline is $n$ times as large as that to slide on a smooth incline, the coefficient of friction is

(a) $\frac{n-1}{n}$ (b) $\frac{n-2}{n^2}$ (c) $\frac{n-4}{n^3}$ (d) $\frac{n^2-1}{n^2}$

Q7. Three identical blocks of masses $m = 2\, \text{kg}$ each are drawn by a force $10.2\, \text{N}$ on a frictionless surface as shown in the figure. What is the tension in the string between the blocks $B$ and $C$?

(a) 9.2 (b) 8 (c) 3.4 (d) 9.8

Q8. A horizontal force of $10\, \text{N}$ is necessary to just hold a block stationary against a wall. The coefficient of friction between the block and the wall is 0.2. The weight of the block is

(a) 20\,\text{N} (b) 50\,\text{N} (c) 100\,\text{N} (d) 2\,\text{N}

Q9. The upper half of an inclined plane with inclination $\theta$ is perfectly smooth, while the lower half is rough. A body starting from rest at the top will again come to rest at friction for the lower half is given by

(a) $2\tan \theta$ (b) $\tan \theta$ (c) $2\sin \theta$ (d) $2\cos \theta$
Q10. The adjacent figure is a part of a horizontally stretched net. Section $AB$ is stretched with a force of $10\, N$.

The tension in the sections $BC$ and $BF$ are
(a) $10\, N, 11\, N$
(b) $10\, N, 6\, N$
(c) $10\, N, 10\, N$
(d) can not be calculated due to insufficient data.

Q11. A particle is moving in a vertical circle. The tension in the string when passing through two positions at angles $30^\circ$ and $60^\circ$ from vertical (lowest positions) are $T_1$ and $T_2$ respectively. Then
(a) $T_1 = T_2$
(b) $T_2 > T_1$
(c) $T_1 > T_2$
(d) Tension in the string always remains the same.

Q12. Two wires $AC$ and $BC$ are tied to a small sphere $C$ of mass $5\, kg$, which revolved at a constant speed $v$ in the horizontal circle of radius $1.6\, m$. Taking $g = 9.8\, m/s^2$, the minimum value of $v$ is

(a) $3.01\, m/s$
(b) $4.01\, m/s$
(c) $3.2\, m/s$
(d) $3.96\, m/s$

Q13. Three equal weights of $3\, kg$ each are hanging on a string passing over a frictionless pulley as shown in the figure. The tension in the string between the masses II and masses III will be
(a) $5\, N$
(b) $6\, N$
(c) $10\, N$
(d) $20\, N$
Q14. A block of mass $m$ is pulled along a horizontal surface by applying a force at an angle $\theta$ with the horizontal. If the block travels with a uniform velocity and has a displaced $d$ and the coefficient of friction is $\mu$, then the work done by the applied force is

(a) $\frac{\mu mgd}{\cos \theta + \mu \sin \theta}$
(b) $\frac{\mu mgd \cos \theta}{\cos \theta + \mu \sin \theta}$
(c) $\frac{\mu mgd \sin \theta}{\cos \theta + \mu \sin \theta}$
(d) $\frac{\mu mgd \cos \theta}{\cos \theta - \mu \sin \theta}$

Q15. Three forces acting on a body are shown in the figure. To have the resultant force only along the $y$-direction, the magnitude of minimum additional force needed along $OX$ is

(a) $0.5 \, N$
(b) $1.5 \, N$
(c) $\frac{\sqrt{3}}{4} \, N$
(d) $\sqrt{3} \, N$

Q16. An object motion is governed by $v = (\cos \omega t) \hat{i} - (\sin \omega t) \hat{j}$. What is the speed of the object?

(a) $\omega$  (b) $2\omega$  (c) $0$  (d) $1$

Q17. A particle motion in the $xy$-plane is governed by its position vector $r = a \sin \omega t \hat{i} + b \cos \omega t \hat{j}$ where $a, b$ and $\omega$ are positive constants and $a > b$. What is the trajectory of the particle?

(a) A parabola  (b) A hyperbola  (c) An ellipse  (d) A circle

Q18. Consider a particle of mass $m$, moving on the plane. Its position vector is given $\vec{r} = a \cos \omega t \hat{i} + b \sin \omega t \hat{j}$ where $a, b, \omega$ are position constants and $a > b$. What is the direction of force?

(a) Always towards the origin  (b) Always away from the origin  (c) Tangential to its trajectory  (d) Unknown
Q19. A block tests on a rough plane, whose inclination $\theta$ to the horizontal can be varied. Which of the following graph indicates how the frictional force $F$ between the block and plane varies as $\theta$ increased?

(a) ![Graph A]
(b) ![Graph B]
(c) ![Graph C]
(d) ![Graph D]

Q20. If a body of mass 45 kg resting on a rough horizontal surface can be just moved by a force of 10 kg weight acting horizontally, then the coefficient of sliding friction is

(a) 4.5  (b) 0.5  (c) 0.45  (d) 0.22

Q21. A block $B$ is resting on a horizontal plate in the $xy$ plane and the coefficient of friction between the block and the plate is $\mu$. The plate begins to move in the $x$-direction and its velocity is $u = bt^2$ being time and $b$ being a constant. At what time will the block start sliding on the plate?

(a) $\frac{ub}{g}$  (b) $\frac{ugb}{2}$  (c) $\frac{ug}{b}$  (d) $\frac{ug}{2b}$
NAT (Numerical Answer Type)

Q22. The force required to accelerate a car of mass 2000 kg from rest to 30 \( \text{m/s} \) in 12 s, if the frictional force between the tyres and the ground is 0.2 \( N/\text{kg} \) is \( \square \) N

Q23. A wooden block having a mass of 1 kg is placed on a table. The block just starts to move, when a force of 10 N is applies at 45\(^\circ\) to the vertical to pull the block. The coefficient of friction between the table and the block, (taking \( g = 10 \text{ m/s}^2 \)) approximate is \( \square \).

Q24. The coefficient of static friction between a box and the flat bed of a track is 0.75. the maximum acceleration the truck can have along the level ground if the box is not to slide is \( \square \text{ m/s}^2 \)

Q25. The coefficient of static friction between block A of mass 2 kg and the table is 0.2. Take \( g = 10 \text{ m/s}^2 \). The maximum mass value of block B so that the two blocks do not move is \( \square \).

Q26. A block of mass 8 kg is placed on a smooth wedge of inclination 30\(^\circ\). The whole system is accelerated horizontally so that the block does not slip on the wedge. If we take \( g = 10 \text{ m/s}^2 \), the force exerted by the wedge on the block will be \( \square \text{ N} \).

Q27. A person of mass 60 kg is inside a lift of mass 940 kg and presses the button on control panel. The lift starts moving upwards with an acceleration 1.0 m/s\(^2\). If \( g = 10 \text{ m/s}^2 \), the tension in the supporting cable is \( \square \text{ N} \).

Q28. A block of mass 4 kg hangs as shown in the figure. If we take \( g = 9.8 \text{ m/s}^2 \), value of \( T_2 \) is \( \square \).

Q29. A block of mass 200 kg is set into motion on a frictionless horizontal surface with the help of a frictionless pulley and rope system as shown in figure. The horizontal force \( F \) that should be applied to produce in the block an acceleration of 1 m/s\(^2\) is \( \square \text{ N} \).
Q30. Two blocks of masses 5 kg and 10 kg are acted on by forces of 3 N and 4 N respectively as shown in the figure.

![Diagram of two blocks with forces](image)

Assuming there is no sliding between the blocks and the ground is smooth, the static friction between the blocks is ________

Q31. Two bodies A and B each of mass 2 kg are connected by a massless spring. A force of 10 N acts on mass B as shown in the figure. At the instant shown body A has acceleration of 1 m/s², the acceleration of B is _______ m/s².

![Diagram of two bodies connected by a spring](image)

Q32. The pulleys and strings shown in the figure are smooth and of negligible mass. For the system to remain in equilibrium the value of θ is ___________ degree.

![Diagram of pulleys and strings](image)

Q33. A horizontal force of 20 N is applied to a block of mass 4 kg resting on a rough horizontal table. If the block does not move on the table, the frictional force from the table on the block is ___________ N. (Take g = 10 m/s²)

Q34. A block slides down an incline at angle 30° with an acceleration $\frac{g}{4}$. The kinetic friction coefficient is ________.
MSQ (Multiple Select Questions)

Q35. The force exerted by the floor of an elevator on the foot of a person standing there is more than the weight of the person if the lift is
(a) going up and slowing down  (b) going up and speeding down
(c) going down and slowing down  (d) going down and speeding up

Q36. If the tension in the cable is supporting an elevator is equal to the weight of the elevator, the elevator may be
(a) going up with increasing speed.  (b) going down with increasing speed
(c) going up with uniform speed  (d) going down with uniform speed

Q37. A block of mass 20 kg is placed gently on an incline plane having an angle of inclination 37°. The coefficient of friction between the block and the incline is 0.1. Take \( g = 10 \text{ m/s}^2 \), \( \sin 37^\circ = \frac{3}{5} \) and \( \cos 37^\circ = \frac{4}{5} \). Then
(a) The frictional force acting on the block is 16 N
(b) The normal force acting on the block is 120 N
(c) A force of static friction acts on the block
(d) The acceleration of the block is 5.2 m/s²

Q38. A block of mass 2.5 kg is kept on a rough horizontal surface. It is found that the block does not slide if a horizontal force less than 15 N is applied to it. Also it is found that it takes 5 s to slide through the first 10 m, if a horizontal force of 15 N is applied and the block is gently pushed to start the motion. If \( g = 10 \text{ m/s}^2 \), then
(a) the coefficient of static friction is 0.50
(b) the coefficient of static friction is 0.60
(c) the coefficient of kinetic friction is 0.52
(d) the coefficient of kinetic friction is 0.40
Q39. A person applies a constant force \( \vec{F} \) on a particle of mass \( m \) and finds that the particle moves in a circle of radius \( r \) with a uniform speed \( v \) as seen from an inertial frame of reference

(a) This is not possible

(b) There are other forces on the particle

(c) The resultant of other forces is \( \frac{mv^2}{r} \) towards the centre.

(d) The resultant of other forces varies in magnitude as well as in direction.

Q40. Two bodies of masses \( m_1 \) and \( m_2 \) are connected by a light string going over a smooth light pulley at the end of a frictionless incline as shown in the figure. The whole system is at rest.

(a) The angle of inline is \( \sin^{-1}\left(\frac{m_1}{m_2}\right) \)

(b) The angle of inline is \( \sin^{-1}\left(\frac{m_2}{m_1}\right) \)

(c) The tension in the string is \( \sin^{-1}\left(\frac{m_2}{m_1}\right) \)

(d) The normal force acting on mass \( m_1 \) is \( m_1g\sqrt{1-\left(\frac{m_2}{m_1}\right)^2} \)

Q41. A system of three blocks are connected together by means of two inextensible strings. Two blocks lie on a horizontal surface and one of them passes over a smooth pulley as shown in the figure. The coefficient of friction between the blocks and the surface is 0.2. Take \( g = 10\, m/s^2 \).

(a) The magnitude of acceleration of the system is \( 0.4\, m/s^2 \)

(b) The magnitude of acceleration of the system is \( 0.8\, m/s^2 \)

(c) The tension in the string connecting the two 1 kg blocks is 2.4 N

(d) The tension in the string attached to 0.5 kg block is 4.8 N
Q42. The friction coefficient between the table and the block shown in the figure is 0.2.

\[ \text{Hence the system can not be in equilibrium. Suppose it moves such that the 15kg block moves downwards with an acceleration } a. \text{ In this case, from the free body diagrams.} \]

(a) The system remains in equilibrium

(b) The system moves with an acceleration of \( \frac{18}{5} \text{ m/s}^2 \)

(c) The tension in the left string is 98N in the left string.

(d) The tension in the right string is 68N

Q43. In the figure shown all the strings are massless, and friction is absent everywhere. Choose the correct option(s).

(a) \( T_1 > T_2 \)

(b) \( T_3 > T_1 \)

(c) \( T_2 > T_1 \)

(d) \( T_2 > T_1 \)

Q44. A block of mass 10kg slides down an inclined surface of inclination 30°. Starting from rest it moves 8m in the first two seconds. Take \( g = 10 \text{ m/s}^2 \).

(a) The acceleration of the block is \( 4 \text{ m/s}^2 \)

(b) The coefficient of kinetic friction is 0.1

(c) The kinetic frictional force acting on the block is 50N

(d) The kinetic frictional force acting on the block is 10N
Solutions
MCQ (Multiple Choice Questions)

Ans. 1: (c)
Solution: Immediately after cutting the force on mass $m$ is $mg$, hence its acceleration is $g$.

Immediately after cutting the spring is unstretched and the force on mass $2m$ is $2mg$, hence its acceleration is $g$.

Ans. 2: (d)
Solution: For the block of mass $M$ to be in equilibrium, the tension in the string $T = Mg$. From the free body diagram of the block $T = Mg$. From the free body diagram of the pulley

$$T = Mg$$

$$F = \sqrt{(Mg + mg)^2 + (Mg)^2} = \sqrt{(M + m)^2 + M^2} g$$

Ans. 3: (a)
Solution: For vertical equilibrium of the block,

$$R = mg + F \sin 60^\circ = \sqrt{3}g + \frac{\sqrt{3}}{2} F$$

For no motion of the block,

$$f \geq F \cos 60^\circ \quad \text{or} \quad \mu R \geq F \cos 60^\circ$$

or

$$\frac{1}{2\sqrt{3}} \left( \sqrt{3}g + \frac{\sqrt{3}}{2} F \right) \geq \frac{F}{2} \quad \text{or} \quad g \geq \frac{F}{2}$$

or

$$F \leq 2g \quad \text{or} \quad F_{\max} = 20 N$$
Ans. 4: (a)

Solution: The block $Q$ oscillates but does not slip on $P$. This indicates acceleration is for both $Q$ and $P$. A force of friction acts between the two blocks but the horizontal plane is frictionless. The system $P-Q$ oscillates with angular frequency,

$$\omega = \sqrt{\frac{k}{m+m}} = \sqrt{\frac{k}{2m}}$$

Maximum acceleration of the system will be

$$a_{\text{max}} = \omega^2 A = \frac{kA}{2m}$$

This acceleration is provided to the lower block by the force of friction.

$$f_{\text{max}} = ma_{\text{max}} = m \frac{kA}{2m} = \frac{kA}{2}$$

Ans. 5: (a)

Solution: As the particle remains stationary under the action of the three forces, so

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0$$

when force $\vec{F}_1$ is removed, the net force left is

$$\vec{F}_2 + \vec{F}_3 = -\vec{F}_1$$

Acceleration $\vec{a} = -\frac{\vec{F}_1}{m}$, hence magnitude of acceleration is $a = \frac{F_1}{m}$.

Ans. 6: (d)

Solution: When there is no friction, the block slides down the inclined plane with acceleration

$$a = g \sin \theta$$

when, there is friction, the downward acceleration of the block is

$$a' = g \left( \sin \theta - \mu \cos \theta \right)$$

As the block slides a distance $d$ in each case, so

$$d = \frac{1}{2} a t^2 = \frac{1}{2} a' t'^2$$
or \( \frac{a}{a'} = \frac{t^2}{t'^2} = \frac{(nt)^2}{t^2} = n^2 \) or \( \frac{g \sin \theta}{g (\sin \theta - \mu \cos \theta)} = n^2 \)

or \( \sin \theta = n^2 \sin \theta - \mu n^2 \cos \theta \) or \( \mu n^2 \cos \theta = (n^2 - 1) \sin \theta \)

or \( \mu = \left(1 - \frac{1}{n^2}\right) \tan \theta = \left(1 - \frac{1}{n^2}\right) \tan 45^\circ \Rightarrow \mu = 1 - \frac{1}{n^2} \)

Ans. 7: (c)

Solution: Free body diagrams for three blocks

Applying Newton's second law, second law,

\[
F - T_1 = ma
\]

\[
T_1 - T_2 = ma
\]

\[
T_2 = ma
\]

Adding the three equations, we get \( F = 3ma \)

or \( a = \frac{F}{3m} \) \( \therefore T_2 = ma = \frac{F}{3} = \frac{102}{3} = 3.4 \text{N} \)

Ans. 8: (d)

Solution: For the block to remain stationary,

\[
R = 10 \text{N}, f = mg
\]

but \( f = \mu R = 0.2 \times 10 = 2 \text{N} \)

Ans. 9: (a)

Solution: As the initial and final speeds of the block are zero,

Work done by gravity = work done by friction

\[
mg \sin \theta \times l = f \times \frac{l}{2}, \text{ where } l \text{ is the length of incline.}
\]

or \( mg \sin \theta = \frac{\mu mg \cos \theta}{2} \) or \( \mu = 2 \tan \theta \)
Ans. 10: (c)
Solution: Let $T_1$ and $T_2$ be the tension in section $BC$ and $BF$ respectively.

$$T_1 \cos 30^\circ = T_2 \cos 30^\circ \Rightarrow T_1 = T_2$$

Also $T_1 \sin 30^\circ + T_2 \sin 30^\circ = 10 \Rightarrow 2T_1 \sin 30^\circ = 10$

$$\Rightarrow 2T_1 \times \frac{1}{2} = 10 \Rightarrow T_1 = 10 \text{ N}$$

Ans. 11: (c)
Solution: At any angle $\theta$ measured from the lowest position

$$T - mg \cos \theta = \frac{mv^2}{r}, \text{ where } r \text{ is the radius of the circle.}$$

or $$T = mg \cos \theta + \frac{mv^2}{r}$$

As $\theta$ increases both $\cos \theta$ and $v$ decrease, hence $T$ decreases.

$\therefore T_1 > T_2$

Ans. 12: (d)
Solution: For body diagram from sphere is shown in the figure.

$$T_1 \cos 30^\circ + T_2 \cos 45^\circ = mg \quad (i)$$

$$T_1 \sin 30^\circ + T_2 \sin 45^\circ = \frac{mv^2}{r} \quad (ii)$$

on solving (i) and (ii) $T_1 = \frac{mg - \frac{mv^2}{r}}{\left(\sqrt{3} - 1\right)}$

$\therefore T_1 > 0 \Rightarrow mg > \frac{mv^2}{r} \Rightarrow v < \sqrt{rg}$

$\therefore v_{\text{max}} = \sqrt{rg} = \sqrt{1.6 \times 9.8} = 3.96 \text{ m/s}$
Ans. 13: (d)
Solution: \( T_1 - mg = ma \)  
(taking the left block as system)
\[ 2mg - T_1 = 2ma \]  
(Taking the two right blocks as system)
\[ mg - T_2 = ma \]  
(Taking the lower block on the right as the system)
\[ T_2 = \frac{2}{3}mg = \frac{2}{3} \times 3 \times 10 = 20 \text{ N} \]

Ans. 14: (b)
Solution: As the block moves with uniform velocity, the resultant force on it should be zero.
\[ R + F \sin \theta = mg \Rightarrow R = mg - F \sin \theta \]
\[ f = \mu R = \mu (mg - F \sin \theta) \]
Also \( F \cos \theta = f \)
or \( F \cos \theta = \mu (mg - F \sin \theta) \)
or \( F (\cos \theta + \mu \sin \theta) = \mu mg \)
\[ F = \frac{\mu mg}{\cos \theta + \mu \sin \theta} \Rightarrow W = Fd \cos \theta = \frac{\mu mg d \cos \theta}{\cos \theta + \mu \sin \theta} \]

Ans. 15: (a)
Solution: \( F + 1 \cos 60^0 + 2 \cos 60^0 = 4 \cos 60^0 \)
or \( F = (4 - 1 - 2) \cos 60^0 = \frac{1}{2} = 0.5 \text{ N} \)

Ans. 16: (d)
Solution: \( \vec{v}_1 = \cos \omega t \hat{i} + \sin \omega t \hat{j} \), \( \vec{v}_2 = \cos \omega t \hat{i} + \sin \omega t \hat{j} \)

\[ \vec{v}_1 \cdot \vec{v}_2 = \left( \cos \omega t \hat{i} - \sin \omega t \hat{j} \right) \cdot \left( \cos \omega t \hat{i} + \sin \omega t \hat{j} \right) = \cos^2 \omega t \sin^2 \omega t = 1 \]

Ans. 17: (c)
Solution: Equation of trajectory is given by \( \vec{r} = x \hat{i} + y \hat{j} = a \sin \omega t \hat{i} + b \cos \omega t \hat{j} \)

\[ x = a \sin \omega t \quad \text{and} \quad y = b \cos \omega t \quad \text{or} \quad \frac{x}{a} = \sin \omega t \quad \text{and} \quad \frac{y}{b} = \cos \omega t \]

Squaring and adding \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = \sin^2 \omega t + \cos^2 \omega t = 1 \), which is an equation of ellipse.
Ans. 18: (a)
Solution: The position vector of the particle is given as \( r = a \cos \omega t \hat{i} + b \sin \omega t \hat{j} \) where \( a, b \) are positive constant with \( a > b \)

Now \( r = a \cos \omega t \hat{i} + b \sin \omega t \hat{j} \)

\[
\Rightarrow \frac{dr}{dt} = -a \omega \cos \omega t \hat{i} + b \omega \cos \omega t \hat{j} \Rightarrow \frac{d^2r}{dt^2} = -a \omega^2 \cos \omega t \hat{i} - b \omega^2 \sin \omega t \hat{j}
\]

\[
\Rightarrow \frac{d^2r}{dt^2} = -\omega^2 \left[ a \cos \omega t \hat{i} + b \sin \omega t \hat{j} \right] \Rightarrow \frac{d^2r}{dt^2} = \omega^2 r
\]

By equation (i)

Force = \( m \times \) acceleration

\[
\Rightarrow \text{Force} = m \left( \frac{d^2r}{dt^2} \right) \Rightarrow F = m(-\omega^2 r) \Rightarrow F = -m\omega^2 r \Rightarrow F \propto (-r)
\]

Negative sign shows that the direction of force is always towards the origin.

Ans. 19: (a)
Solution: Friction force, \( F = \mu mg \sin \theta \Rightarrow F \propto \sin \theta \)

Thus, correct graph is (a)

Ans. 20: (d)
Solution: The force \( F = \mu mg \)

\[
\Rightarrow 10 g = \mu \times 45 g
\]

\[
\Rightarrow \mu = \frac{10}{45} = 0.22
\]

Ans. 21: (d)
Solution: The block \( B \) starts to move with velocity \( u = bt^2 \) so, frictional force applied on this is \( \mu R \) in \( -x \) axis direction as shown.

For starting sliding \( F_{\min} \geq \) limiting friction

\[
\Rightarrow m \frac{du}{dt} \geq \mu R \Rightarrow m \frac{d}{dt} \left( bt^2 \right) \geq \mu mg
\]

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\[ m(2bt) \geq \mu mg \Rightarrow t \geq \frac{\mu mg}{2bm} \]
\[ t \geq \frac{\mu g}{2b} \]

NAT (Numerical Answer Type)

Ans. 22: 5400 N

Solution: \( m = 2000 \text{kg}, u = 0, v = 30 \text{ m/s}, t = 12s \)

\[ v = u + at \Rightarrow 30 = 0 + a \times 12 \Rightarrow a = 2.5 \text{ m/s}^2 \]

The force required for acceleration = \( 2.5 \times 2000 = 5000 \text{ N} \)

Friction force = \( \mu mg = 0.2 \times 2000 = 400 \text{ N} \)

Total force = 5000 + 400 = 5400 N

Ans. 23: 0.7

Solution: Resolving the forces and getting their horizontal and vertical component

\[ mg = R + F \cos \theta \] (i)

Frictional force = \( F \sin \theta \) (ii)

By equations (ii), we get

\[ \Rightarrow \mu mg = F \sin 45^\circ \]

\[ \Rightarrow \mu = \frac{F \sin 45^\circ}{mg} = \frac{10 \times \frac{1}{\sqrt{2}}}{1 \times 10} = \frac{1}{\sqrt{2}} = 0.707 \]

Ans. 24: 7.35 m/s^2

Solution: Writing the equation of motion. If \( a \) is acceleration. Then

\[ ma = F \Rightarrow ma = \mu R \Rightarrow R = mg \]

So, \( ma = \mu mg \Rightarrow a = \frac{m \mu g}{m} \Rightarrow a = \mu g \)

Here, \( \mu = 0.75, g = 9.8 \text{ m/s}^2 \)

So, acceleration \( a = 0.75 \times 9.8 = 7.35 \text{ m/s}^2 \)
Ans. 25: 0.4 kg

Solution: For the equilibrium of block $A$, the force of friction must balance the tension in the string

\[ \mu m_A g = T \]  \hspace{1cm} (i)

For the equilibrium of block $B$, the tension in the string must be equal to its weight

\[ m_B g = T \]

Hence $m_B g = \mu m_A g$

or $m_B = \mu m_A = 0.2 \times 2 = 0.4$ kg

Ans. 26: $\frac{160}{\sqrt{3}}$ or 92.48

Solution: Let the whole system is accelerated horizontally with an acceleration $a$.

For the block $ma \cos \theta = mg \sin \theta$

or $a = g \tan \theta$

In the direction normal to the plane

\[ R = mg \cos \theta + ma \sin \theta = mg \cos \theta + m \cdot \frac{g \sin \theta}{\cos \theta} \cdot \sin \theta \]

\[ R = mg \left( \cos^2 \theta + \sin^2 \theta \right) = mg \cos \theta \]

\[ R = \frac{8 \times 10}{\cos 30^\circ} = \frac{8 \times 10 \times 2}{\sqrt{3}} \text{ or } R = \frac{160}{\sqrt{3}} N = 92.48 N \]

Ans. 27: 11000

Solution: From the free body diagram,

\[ T - (M_m + M_i)g = (M_m + M_i)a \]

where $M_m$ and $M_i$ are the masses of the man and the lift respectively.

\[ T = (M_m + M_i)(g + a) = (940 + 60)(10 + 1) = 11000 N \]
Ans. 28: 67.8

Solution: Resolve the tension $T_1$ along horizontal and vertical directions and as the body is in equilibrium

\[ T_1 \sin 30^\circ = 4 \times 9.8 \]
\[ T_1 \cos 30^\circ = T_2 \]

so
\[ T_1 = \frac{4 \times 9.8}{\sin 30^\circ} = 78.4 \text{ N} \]

\[ T_2 = T_1 \cos 30^\circ = \frac{78.4 \times \sqrt{3}}{2} = 39.2\sqrt{3} = 67.8 \text{ N} \]

Ans. 29: 100

Solution: As shown in the force body diagram, when force $F$ is applied at the end of the string, the tension in the lower part of the string is also $F$.

\[ a = 1 \text{ m/s}^2 \]

If $T$ is the tension in the string connecting the pulley and the block, then
\[ T = 2F \quad \text{or} \quad T = ma = (200)(1) = 200 \text{ N} \]
\[ \therefore 2F = 200 \text{ N} \quad \text{or} \quad F = 100 \text{ N} \]

Ans. 30: 0.67

Solution: The free body diagram of the blocks is

From Newton’s second law,
\[ 3 - f = 5a_1 \quad \text{(i)} \]
\[ 4 + f = 10a_2 \quad \text{(ii)} \]

Since the friction is static $a_1 = a_2$

Hence using (i) and (ii)
\[
\frac{3 - f}{5} = \frac{4 + f}{10}
\]

this gives \( f = \frac{2}{3} = 0.67 \)

Ans. 31: 4

Solution: Let \( T \) be the tension in the string. Then the equation of motion for \( A \) is

\[ T = 2.1 = 2N \]

Let \( a \) be the acceleration of \( B \). Then

\[ 2a = 10 - T \Rightarrow a = 4 \text{ m/s}^2 \]

Ans. 32: 45

Solution: The free body diagrams of masses \( \sqrt{2} m \) are shown

For the equilibrium of mass \( m \)

\[ mg = T \]

For the equilibrium of mass \( \sqrt{2} m \)

or \( \sqrt{2} mg = 2T \cos \theta \)

or \( \sqrt{2} \times T = 2T \cos \theta \)

\[ \therefore \theta = 45^0 \]

Ans. 33: 20

Solution: The free body diagram is shown in the figure.

As the block is at rest, these forces should add up to zero. Taking horizontal and vertical components, \( f = 20N \)
Ans. 34: \( \frac{1}{2\sqrt{3}} \)

Solution: Free body diagram is shown below

Taking components parallel to the incline and writing Newton’s second law

\[
m g \sin 30^\circ - f = \frac{mg}{4} \Rightarrow f = \frac{mg}{4}
\]

There is no acceleration perpendicular to the incline. Hence

\[
N = mg \cos 30^\circ = mg \frac{\sqrt{3}}{2}
\]

as the block is slipping on the incline, friction is

\[f = \mu_k N\]

so

\[\mu_k = \frac{f}{N} = \frac{mg}{4mg} \frac{\sqrt{3}}{2} \quad \text{or} \quad \mu_k = \frac{1}{2\sqrt{3}}\]

MSQ (Multiple Select Questions)

Ans. 35: (b) and (c)

Solution: In case (a) acceleration is downwards.

In case (b) acceleration is upwards

In case (c) acceleration is upwards

In case (d) acceleration is downwards

\[N = m(g + a)\], hence the force exerted by the floor is more than the true weight.

Ans. 36: (c) and (d)

Solution: We have \( T - mg = ma \) or \( T = m(g + a) \) for upwards acceleration

For downwards acceleration \( mg - T = ma \) or \( T = m(g - a) \)

For uniform speed up or down

\[T = mg\], since acceleration of the elevator is zero.
Ans. 37: (a), (b) and (d)

Solution: The normal force acting on the block is

\[ N = mg \cos 31^\circ = 20 \times 10 \times \frac{4}{5} = 160 \, N \]

The maximum value of frictional force acting on the block is

\[ f_{\text{max}} = \mu N = 0.1 \times 160 = 16 \, N \]

The component of weight pulling the block down the incline

\[ = mg \sin 37^\circ = 20 \times 10 \times \frac{3}{5} = 120 \, N \]

Since this component is greater than the maximum value of frictional force, the block slides down the incline and the frictional force is at its maximum value.

Hence \( f = f_{\text{max}} = 16 \, N \)

From Newton’s second law, \( mg \sin 37^\circ - f = ma \)

or \[ 20 \times 10 \times \frac{3}{5} - 16 = 20 \times a \Rightarrow 120 - 16 = 20a \Rightarrow a = 5.2 \, m/s^2 \]

Ans. 38: (b) and (c)

Solution: Free body diagram of block is shown

The maximum value of static friction force is 15 \( N \) since the block does not slide up to this force.

Hence \( \mu_s N = 15 \)

or \[ \mu_s = \frac{15}{N} = \frac{15}{Mg} = \frac{15}{2.5 \times 10} \]

\[ \therefore \mu_s = 0.60 \]

when the block is gently pushed to start the motion, kinetic friction acts between the block and the surface. Since the block takes 5 seconds to slide through the first 10 \( m \), the acceleration \( a \) given by

\[ 10 = \frac{1}{2} \times a \times (5)^2 \quad \text{or} \quad a = 0.8 \, m/s^2 \]
Applying Newton’s second law

\[ F = -\mu_k M g = Ma \]

or \( \mu_k = \frac{F}{M g} \)

or \( \mu_k = \frac{15N - (2.5)(0.8)}{(2.5)(10)} \) or \( \mu_k = 0.52 \)

Ans. 39: (b) and (d)

Solution: If the net force acting on the particle is \( \frac{m v^2}{r} \) towards the centre the particle moves in a circle of radius \( r \) with uniform speed. A single constant force can not always point towards the centre. Hence there are other forces acting on the particle such that the resultant of \( \vec{F} \) and there other forces points towards the centre.

Take two positions of the circle. From the figure we see that \( \vec{F}_{\text{other}} \) must vary in magnitude as well as in direction.

Ans. 40: (b) and (d)

Solution: Free body diagrams of two masses as shown

Let the angle of incline be \( \theta \). From the free body diagrams

For mass \( m_1 \)

\( N = m_1 g \cos \theta \) \hspace{1cm} (i)

and \( T = m_1 g \sin \theta \) \hspace{1cm} (ii)

For mass \( m_2 \)

\( T = m_2 g \) \hspace{1cm} (iii)

From (ii) and (iii) \( \sin \theta = \frac{m_2}{m_1} \) or \( \theta = \sin^{-1} \frac{m_2}{m_1} \)

From equation (ii) \( T = m_1 g \sin^{-1} \left( \frac{m_2}{m_1} \right) \)
From equation (i) \[ N = m_1 g \sqrt{1 - \left( \frac{m_2}{m_1} \right)^2} \]

Ans. 41: (a), (b) and (d)

Solution: Let \( a \) be the magnitude of the acceleration of the system.

Taking the left 1\( \text{kg} \) block as if system

\[ T_1 - f = 1.0 \times a \]

or \[ T_1 - 0.2 \times 1.0 \times 10 = a \]

or \[ T_1 - 2 = a \]  

(i)

Taking right 10\( \text{kg} \) block as if system

\[ T_2 - T_1 - f = 1.0 \times a \]

or \[ T_2 - T_1 - 0.2 \times 1.0 \times 10 = a \]  

(ii)

Taking 0.5\( \text{kg} \) block as if system

\[ 0.5 \times 10 - T_2 = 0.5 a \]

or \[ 5 - T_2 = 0.5 a \]  

(iii)

putting the values of \( T_1 \) from (i) and \( T_2 \) from (iii) in equation (ii)

\[ (5 - 0.5 a) - (2 + a) - 2 = a \]

or \[ 2.5 a = 1 \]  

or \[ a = 0.4 \text{ m/s}^2 \]

From equation (i) \[ T_1 = 2 + a = 2.4 \text{ m/s}^2 \]

From equation (iii) \[ T_2 = 5 - 0.5 \times 0.4 = 4.8 \text{ m/s}^2 \]

Ans. 42: (b), (c) and (d)

Solution: Suppose the system is in equilibrium then the regained value of frictional force on the 5\( \text{kg} \) block is

\[ T_2 + f - T_1 = 0 \]

or \[ f = T_1 - T_2 \]
but in the case
\[
T_1 = 15 \times 10 = 150N \quad \text{and} \quad T_2 = 5 \times 10 = 50N
\]
hence \( f = 150 - 50 = 100N \)

But in the maximum possible frictional force acting on the middle 5.0 kg mass is
\[
f_{\text{max}} = 0.2 \times 5 \times 10 = 10N
\]
hence the system can not be in equilibrium. Suppose it moves such that the 15 kg block moves downwards with an acceleration \( a \). In this case, from the free body diagrams

\[
15 \times 10 - T_1 = 15a
\]
or
\[
150 - T_1 = 15a \quad (i)
\]

\[
T_1 - T_2 - f_k = 5a
\]
or
\[
T_1 - T_2 - 0.2 \times 5 \times 10 = 5a \quad (ii)
\]
or
\[
T_1 - T_2 - 10 = 5a \quad (ii)
\]

\[
T_2 - 5 \times 10 = 5a
\]
or
\[
T_2 - 50 = 5a \quad (iii)
\]

Putting \( T_1 \) and \( T_2 \) from (i) and (iii) respectively in (ii):

\[
(150 - 15a) - (59 - 50) - 10 = 5a \Rightarrow 25a = 90 \quad \text{or} \quad a = \frac{18}{5} m/s^2
\]

\[
T_2 = 50 + 59 = 50 + 5 \times \frac{18}{5} = 68N
\]

\[
T_1 = 150 - 15a = 150 - 15 \times \frac{18}{5} = 96N
\]
Ans. 43: (b), (c) and (d)

Solution: Let the system moves clockwise with acceleration \( a \). Hence free body diagrams of different blocks.

\[
\begin{align*}
T_2 & \quad 1g \\
T_3 & \quad 2g \\
\text{Left 1kg mass} & \\
T_2 & \quad 1g \\
T_3 & \quad 2g \\
\text{Left 2kg mass} & \\
T_2 & \quad T_1 \\
\text{Right 2kg mass} & \\
T_1 & \quad 1g \\
\text{Right 1kg mass}
\end{align*}
\]

Hence
\[
\begin{align*}
T_2 - 10 - T_3 &= a & \text{(i)} \\
T_3 - 20 &= 2a & \text{(ii)} \\
T_1 + 20 - T_2 &= 2a & \text{(iii)} \\
10 - T_1 &= a & \text{(iv)}
\end{align*}
\]

Solving these equations gives \( a = 0 \).

Hence, \( T_1 = 10N, T_2 = 30N, T_3 = 20N \)

Ans. 44: (a), (b) and (d)

Solution: The free body diagram is shown.

From the relation
\[
s = ut + \frac{1}{2}at^2 \Rightarrow 8 = 0 + \frac{1}{2} \times a \times 4 \Rightarrow a = 4 \text{ m/s}^2
\]

From Newton’s second law
\[
mg \sin 30^\circ - f_k = ma
\]

or \( f_k = \frac{mg}{2} - ma \) or \( f_k = \frac{10 \times 10}{2} - 10 \times 4 = 10N \)

\[ \therefore \text{ From the relation } \]

\[ f_k = \mu_k mg \text{ or } \mu_k = \frac{f_k}{mg} = \frac{10}{10 \times 10} = 0.1 \]
2. Energy and Momentum

Conservation of energy:- The energy can neither will created nor destroy it can be transformer from one form to another form.

2.1 Different Form of Energy used in Mechanics

2.1.1 Kinetic energy: The energy which is responsible for motion of the particle. If particle of mass \( m \) moving with velocity \( \vec{v} \) then kinetic energy is given by \( K = \frac{1}{2} m (\vec{v} \cdot \vec{v}) \).

Kinetic energy is Cartesian coordinate \( K = \frac{1}{2} m (x^2 + y^2 + z^2) \)

Kinetic energy in cylindrical coordinate \( K = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + z^2) \)

Kinetic energy in spherical coordinate \( K = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \varphi \dot{\varphi}^2) \)

2.1.2 Potential energy: \( U \) – the energy which is required to perform the work is known as potential energy \( U \). Hence force is defined as \( F = -\frac{\partial U}{\partial r} \) then potential energy \( U_b - U_a = \int_a^b F \cdot d\vec{r} \) for the conservative force one can say potential energy is negative integral of the force .one can say in another way change in potential energy with respect to position is cause of force \( F \).

There is different type of potential energy for example electrostatic potential energy, gravitation potential energy , stored energy in spring , mass energy in relativistic mechanics .

Total energy \( E \) is sum of kinetic energy \( K \) and potential energy \( U \). So total energy \( E = K + U \)

Energy diagram …..

2.2 The work-energy Theorem

The work-energy Theorem is given by change in kinetic energy is equal to work done by the system.

The quantity \( \frac{1}{2} m v^2 \) is called the kinetic energy \( K \), and the change in kinetic energy can be written as \( K_b - K_a \). The integral \( \int_a^b F \cdot d\vec{r} \) is called the work \( W_{ba} \) done by the force \( F \) on the particle.
From work energy theorem \( W_{ba} = K_b - K_a \).

This result is the general statement of the work-energy theorem which we met in restricted form in our discussion of one dimensional motion.

The work \( \Delta W \) done by a force \( F \) in a small displacement \( \Delta r \) is

\[
\Delta W = F \cdot \Delta r = F \cos \theta \Delta r = F_1 \Delta r,
\]

where \( F_1 = F \cos \theta \) is the component of \( F \) along the direction of \( \Delta r \).

The component of \( F \) perpendicular to \( \Delta r \) does no work. For a finite displacement from \( r_a \) to \( r_b \), the work on the particle, \( \int F \cdot dr \), is the sum of the contribution \( \Delta W = F \Delta r \) from each segment of the path, in the limit where the size of each segment approaches zero.

In the work-energy theorem, \( W_{ba} = K_b - K_a \), \( W_{ba} \) is the work done on the particle by the total force \( F \). If \( F \) is the sum of several forces \( F_i \), we can write

\[
W_{ba} = \sum_i (W_i)_{ba} = K_b - K_a,
\]

where \( (W_i)_{ba} = \int_{r_a}^{r_b} F_i \cdot dr \) is the work done by the \( i \)th force \( F_i \).

Work energy theorem in one dimension

we demonstrated the formal procedure for integrating Newton’s second law with respect to position. The result was

\[
\frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = \int_{x_a}^{x_b} F(x) \, dx,
\]

which we now wish to interpret in physical terms.

The quantity \( \frac{1}{2}mv^2 \) is called the kinetic energy \( K \) and the left hand side can be written \( K_b - K_a \).

The integral \( \int_{x_a}^{x_b} F(x) \, dx \) is called the work \( W_{ba} = K_b - K_a \).

This result is known as the work-energy theorem or, more precisely, the work-energy theorem in one dimension. (We shall shortly see a more general statement). The unit of work and energy in the SI system is the Joule (J).
1. $J = 1 \text{kgm}^2/\text{s}^2$. The unit of work and energy in the cgs system is the erg:

$$1 \text{erg} = 1 \text{gcm}^2/\text{s}^2 = 10^{-7} J$$

The unit of work in the English system is the foot-pound: $1 \text{ft} \cdot \text{lb} \approx 1.336 \ J$.

Conservative forces **conservative forces** if $\vec{F}$ is a force and it is given that $\nabla \times \vec{F} = 0$ then force is said to be non conservative. The work done due to non conservative forces is path independent.

### 2.3 Work Done by a Uniform Force

The case of a uniform force is particularly simple. Here is how to find the work done by a force.

$F = F_0 \hat{n}$ where $F_0$ is a constant and $\hat{n}$ is a unit vector in some direction, as the particle moves from $r_a$ to $r_b$ along some arbitrary path. All the steps are put in to make the procedure clear, but with any practice this problem can be solved by inspection.

$$W_{ba} = \frac{1}{2} \int_{a}^{b} F \cdot dr = \frac{1}{2} \int_{a}^{b} F_0 \hat{n} \cdot dr = F_0 \hat{n} \cdot \int_{a}^{b} dr$$

$$= F_0 \hat{n} \left( \int_{a}^{b} i \int_{a}^{b} j \int_{a}^{b} k \right) dx + \int_{a}^{b} dy + \int_{a}^{b} dz$$

$$= F_0 \hat{n} \left( \int_{a}^{b} (x_b - x_a) + \int_{a}^{b} (y_b - y_a) + \int_{a}^{b} (z_b - z_a) \right)$$

$$= F_0 \hat{n} \cdot (r_b - r_a) = F_0 \cos \theta |r_b - r_a|$$

For a constant force the work depends only on the net displacement. $r_b - r_a$, not on the path followed. This is not generally the case, but it holds true for an important group of forces.

### 2.4 Non conservative forces

If $\vec{F}$ is force and it is given that $\nabla \times \vec{F} \neq 0$ then force is said to be non conservative. The work done due to non conservative forces is path dependent.

For non conservative force $\vec{F}$ work energy theorem is given by $E_b - E_a = \int_{a}^{b} \vec{F} \cdot d\vec{r}$

Where $E_b$ and $E_a$ is total energy at point $b$ and $a$ respectively.
Example: Consider a mass $M$ attached to a spring with spring constant $k$. Using the coordinate $x$ measured from the equilibrium point $x_0$ solve the Equation of motion for Simple Harmonic Motion with help of work energy theorem.

Solution: Consider a mass $M$ attached to a spring. Using the coordinate $x$ measured from the equilibrium point, the spring force is $F = -kx$.

\[
\frac{1}{2} Mv^2 - \frac{1}{2} Mv_0^2 = -k|\dot{x}|_0 x \, dx = - \frac{1}{2} kx^2 + \frac{1}{2} kx_0^2
\]

In order to find $x$ and $v$, we must know their values at some time $t_0$. Let us consider the case where at $t=0$ the mass is released from rest, $v_0 = 0$, at a distance $x_0$ from the origin. Then

\[
v^2 = -\frac{k}{M} x^2 + \frac{k}{M} x_0^2 \quad \text{and} \quad \frac{dx}{dt} = \sqrt{\frac{k}{M} \sqrt{x_0^2 - x^2}}
\]

Separating the variables gives.

\[
\int_{x_0}^{x} \frac{dx}{\sqrt{x_0^2 - x^2}} = \frac{k}{M} \int_{0}^{t} dt = \frac{k}{M} t \Rightarrow \sin^{-1} \frac{x}{x_0} = \sqrt{\frac{k}{M} t}
\]

Example A mass $m$ is shot vertically upward from the surface of the earth with initial speed $v_0$. Assuming that the only force is gravity, find its maximum altitude and the minimum value of $v_0$ for the mass to escape the earth completely.

Solution: The force on $m$ is $F = -\frac{GMm}{r^2}$.

The problem is one dimensional in the variable $r$ and it is simple to find the kinetic energy at distance $r$ by the work-energy theorem.
Let the particle start at \( r = R \), with initial velocity \( v_o \).

\[
K(r) - K(R) = \int_{R}^{r} F(r) \, dr = -GM_{e}m_{e} \frac{dr}{r^{2}}
\]

or

\[
\frac{1}{2}mv(r)^{2} - \frac{1}{2}mv_{o}^{2} = GM_{e}\left(\frac{1}{r} - \frac{1}{R}\right)
\]

We can immediately find the maximum height of \( m \). At the highest point, \( v(r) = 0 \) and we have

\[
v_{o}^{2} = 2GM_{e}\left(\frac{1}{R} - \frac{1}{r_{max}}\right).
\]

It is a good idea to introduce known familiar constants whenever possible. For example, since

\[
g = \frac{GM}{R^{2}},
\]

we can write

\[
v_{o}^{2} = 2gR_{e}\left(\frac{1}{R_{e}} - \frac{1}{r_{max}}\right) = 2gR_{e}\left(1 - \frac{R}{r_{max}}\right) \text{ or } r_{max} = \frac{R_{e}}{1 - \frac{v_{o}^{2}}{2gR_{e}}}
\]

The escape velocity from the earth is the initial velocity needed to move \( r_{max} \) to infinity. The escape velocity is therefore \( v_{escape} = \sqrt{2gR_{e}} = \sqrt{2 \times 9.8 \times 6.4 \times 10^{6}} = 1.1 \times 10^{4} \text{ m/s} \)

Example: A small bar \( A \) resting on a smooth horizontal plane is attached by threads to a point \( P \) and by means of a weightless pulley, to a weight \( B \) possessing the same mass as the bar itself. Besides the bar is also attached to a point \( O \) by means of a light non-deformed spring of length \( l_{o} \) cm and stiffness \( k \times m \) mg/cm, where \( m \) is the mass of the bar. The thread \( PA \) having been burned, the bar starts moving. Find its velocity at the moment when it is breaking off the plane.
Solution: When bay $A$ breaking off the plane then normal reaction will be zero, then

**Force equation:**

$$kl_0[\csc \theta - 1] \sin \theta = mg$$

$$kl_0[1 - \sin \theta] = mg$$

$$\sin \theta = \frac{kl_0 - mg}{kl_0} = 1 - \frac{mg}{kl_0}$$

**Energy equation:** from work energy theorem

Loss in P.E $(mg) =$ Gain in K.E. of both block + spring energy.

$$mg l_0 \cot \theta = \left( \frac{1}{2} m v^2 \right) + \left( \frac{1}{2} m v^2 \right) + \frac{1}{2} R l_0 (\csc \theta - 1)^2$$

... (ii)

from (i) and (ii): $v = \sqrt{\frac{19 gl_0}{32}}$

**Example:** A block of mass $M$ slides down a plane of angle $\theta$. The problem is to find the speed of the block after it has descended through height $h$, assuming that it starts from rest and that the coefficient of friction $\mu$ is constant.

Initially the block is at rest at height $h$; finally the block is moving with speed $v$ at height 0. Hence

$\text{Solution: }$ The potential energy, kinetic energy and total energy at point $a$ and $b$ respectively

$$U_a = Mgh, \ U_b = 0 \ K_a = 0, \ K_b = \frac{1}{2} Mv^2, \ E_a = Mgh, \ E_b = \frac{1}{2} Mv^2$$

The non-conservative force is $f = \mu N = \mu Mg \cos \theta$. Hence, conservative work is

$$W_{nc}^{ba} = \int_a^b f \cdot dr = -fs,$$
where \( s \) is the distance the block slides. The negative sign arises because the direction of \( f \) is always opposite to the displacement, so that \( f \cdot dr = -f \cdot dr \). Using \( s = h / \sin \theta \), we have

\[
W_{\text{ns}}^{\text{nc}} = -\mu M g \cos \theta \frac{h}{\sin \theta} = -\mu \cot \theta M g h.
\]

The energy equation \( E_s - E_a = W_{\text{ns}}^{\text{nc}} \) becomes

\[
\frac{1}{2} M v^2 - M g h = -\mu \cot \theta M g h,
\]

which gives \( v = \left[ 2 (1 - \mu \cot \theta) g h \right]^{1/2} \).

Example prove that for \( F = A \left( x \hat{i} + y^2 \hat{j} \right) \) is non conservative and

(b) Consider the integral \( W = \int \overline{F} \cdot dr \) from \((0,0)\) to \((0,1)\), first along path 1 and then along path 2, as shown in the figure. Check whether is path independent or not.

(a) \( \overline{\nabla} \times \overline{F} = -x \hat{k} \neq 0 \). So force is non conservative.

(b) The force \( F \) has no physical significance, but the example illustrates the properties of non conservative forces. Since the segments of each path lie along a coordinate axis, it is particularly simple to evaluate the integrals. For path 1 we have

\[
\int_{a}^{b} F \cdot dr = \int_{a}^{b} F \cdot dx + \int_{a}^{b} F \cdot dy = A
\]

Along segment \( dr = dx \hat{i} \), \( F \cdot dr = F \cdot dx = A \). Since \( y = 0 \) along the line of this integration,

\[
\int_{a}^{b} F \cdot dy = 0.
\]

Similarly, for path \( b \), \( \int_{b}^{c} F \cdot dy = A \). While for path \( c \),

\[
\int_{c}^{d} F \cdot dx = A
\]

Thus \( \int_{a}^{d} F \cdot dr = \frac{A}{3} \).
2.5 Energy Diagrams

We can often find the most interesting features of the motion of a one dimensional system by using an energy diagram, in which the total energy $E$ and the potential energy $U$ are plotted as functions of position. The kinetic energy $K = E - U$ is easily found by inspection. Since kinetic energy can never be negative, the motion of the system is constrained to regions where $U \leq E$.

2.5.1 Energy Diagram of Bounded Motion

Here is the energy diagram for a harmonic oscillator. The potential energy $U = \frac{1}{2} kx^2$ is a parabola centered at the origin. Since the total energy is constant for a conservative system, $E$ is represented by a horizontal straight line. Motion is limited to the shaded region where $E \geq U$; the limits of the motion, $x_1$ and $x_2$ in the sketch, are sometimes called the turn points.

Here is what the diagram tells us. The kinetic energy, $K = E - U$ is greatest at the origin. As the particle flies past the origin in either direction, it is slowed by the spring and comes to a complete rest at one of the turning points $x_1, x_2$. The particle then moves toward the origin with increasing kinetic energy and the cycle is repeated.

The harmonic oscillator provides a good example of bounded motion. As $E$ increases, the turning points move farther and farther off, but the particle can never move away freely. If $E$ decreased, the amplitude of motion decreases, until finally for $E = 0$ the particle lies at rest at $x = 0$.

2.5.2 Energy Diagram of Unbounded Motion

the pot $U = A/r$, where $A$ is positive. There is a distance of closest approach $r_{\text{min}}$, as shown in the diagram, but the motion is not bounded for large $r$ since $U$ decreases with distance. If the particle is shot toward the origin, it gradually loses kinetic energy until it comes momentarily to
rest at \( r_{\text{min}} \). The motion then reverses and the particle moves out toward infinity. The final and initial speeds at any point are identical; the collision merely reverses the velocity.

![Energy diagram](image)

For positive energy, \( E > 0 \), the motion is unbounded, and the atoms are free to fly apart. As the diagram indicates, the distance of closest approach, \( r_{\text{min}} \), does not change appreciably as \( E \) is increased. The steep slope of the potential energy curve at small \( r \), the kinetic energy will zero at \( r_{\text{min}} \) and as \( r \) increases the potential energy decrease but kinetic energy \( KE = E - U \) will increases sharply.

**Phase Curve:** The curve between momentum and its conjugate momentum for given value of energy is known as phase curve.

### 2.6 Stability and Unstability in One Dimension

Any potential can be function of and generalized velocity \( U = U(x, \dot{x}, t) \) the equilibrium point is defined where total external force on the system is zero i.e for any co-ordinate say \( x \) is said to be equilibrium point if \( \frac{\partial U}{\partial x} = 0 \) at \( x = x_0 \)

If \( x_0 \) is maxima or (local maxima) i.e. \( \frac{\partial^2 U}{\partial x^2} \bigg|_{x=x_0} \leq 0 \) then it is unstable equilibrium point. Unstable equilibrium point always behave like repulsive point.

And if \( x_0 \) is minima or (local minima) i.e. \( \frac{\partial^2 U}{\partial x^2} \bigg|_{x=x_0} \geq 0 \) then it is stable equilibrium point. Stable equilibrium always behaves as attractive point. The phase portrait is a geometrical representation of trajectories of dynamical system in phase plane which is defined by generalized coordinate and its conjugate generalized momentum. The phase trajectory about unstable point is mainly in
bounded but phase trajectory about stable point is bounded and motion is defined as small oscillation.

Example: If potential in one dimension is given by \( V(x) = -\frac{x^2}{2} + \frac{x^4}{4} \) then

(a) Find the point where potential is zero

(b) Find the equilibrium point.

(c) Find the stable and unstable equilibrium

(e) Draw phase curve ie \( p(x) \) Vs \( x \) for given energy \( E \).

Solution:

(a) \( V(x) = 0 \Rightarrow -\frac{x^2}{2} + \frac{x^4}{4} = 0 \Rightarrow x = 0, \pm \sqrt{2}, -\sqrt{2} \)

(b) For equilibrium point \( \frac{\partial V}{\partial x} = 0 \Rightarrow -x + x^3 = 0 \) so there is three equilibrium points

\[ x_1 = 0, \quad x_2 = 1, \quad x_3 = -1 \]

(c) For discussion of stability and instability we must found \( \frac{\partial^2 V}{\partial x^2} = -1 + 3x^2 \) for stable equilibrium \( \frac{\partial^2 V}{\partial x^2} > 0 \) at \( x_2 = 1 \) and \( x_3 = -1 \) the value of \( \frac{\partial^2 V}{\partial x^2} = 2 \) which is greater than 0 for unstable point \( \frac{\partial^2 V}{\partial x^2} < 0 \) at \( x_1 = 0 \) the value of \( \frac{\partial^2 V}{\partial x^2} = -1 \) which is lesser than 0 so it is unstable point.

(d) \( p(x) \) vs \( x \)

To plot phase curve first one should plot potential \( (V \text{ vs } x) \), then on same axis one should plot momentum plot with common \( x \) axis.

We can check how momentum is changing with position with keeping in mind how potential is changing.

With position .for example if potential is increasing then kinetic will decreasing or vice versa because total energy will always remain constant. One should plot the phase curve for different range of energy. For example in this potential there are three range of energy.

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Case 1- If $-\frac{1}{4} < E < 0$ the particle has motion about stable equilibrium point $x = 1, -1$ the motion is bounded.

Case 2- If $0 < E < \infty$ the particle has motion about unstable equilibrium point $x = 0$ the motion is bounded.

Case 3- At $E = 0$ the particle can be landed exactly at unstable equilibrium which is nature of transition from case-1 to case-2.

2.6.1 Small Oscillation:

Let us assume the potential $U(x)$ has stable equilibrium point at $x = x_0$ then \( \frac{\partial U}{\partial x} \bigg|_{x=x_0} = 0 \) and \( \frac{\partial^2 U}{\partial x^2} \bigg|_{x=x_0} \geq 0 \)

one will do Taylor expansion of $U(x)$ about $x = x_0$, then

\[
U(x) = U(x_0) + \frac{\partial U}{\partial x} \bigg|_{x=x_0} (x - x_0) + \frac{1}{2} \frac{\partial^2 U}{\partial x^2} \bigg|_{x=x_0} (x - x_0)^2 + \text{order}(x - x_0)^3 \ldots
\]

If term $(x - x_0)^2$ is small then higher order term can be neglected then potential energy is equivalent to $U(x) = U(x_0) + \frac{1}{2} \frac{\partial^2 U}{\partial x^2} \bigg|_{x=x_0} (x - x_0)^2$ because \( \frac{\partial U}{\partial x} \bigg|_{x=x_0} = 0 \)

So force is equal to $F = -\frac{\partial U}{\partial x}$, $F = -\frac{\partial^2 U}{\partial x^2} \bigg|_{x=x_0} (x - x_0)$ . Hence $F \propto -(x - x_0)$ then motion is small oscillation the angular frequency is given by $\omega = \sqrt{\frac{\frac{\partial^2 U}{\partial x^2} \bigg|_{x=x_0}}{m}}$ where $m$ is mass of the particle . the term \( \frac{\partial^2 U}{\partial x^2} \bigg|_{x=x_0} \) is identified as spring constant.
Example: If particle of mass \( m \) is interact with potential \( ax^2 + \frac{b}{x^2} \) then what is frequency of the oscillation? (Assume oscillation is small)?

Solution: \( V(x) = ax^2 + \frac{b}{x^2} \)

For equilibrium point \( \frac{\partial V}{\partial x} = 0 \)

\[
2ax - \frac{2b}{x^3} = 0 \implies ax^4 - b = 0 \implies x_0 = \left( \frac{b}{a} \right)^{1/4}
\]

\[
k = \left. \frac{\partial^2 V}{\partial x^2} \right|_{x=x_0} = 2a + \frac{2.3b}{x^4}
\]

Put value of \( x_0 = \left( \frac{b}{a} \right)^{1/4} \)

\[
k = 2a + \frac{6b \times a}{b} = 8a \implies \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{8a}{m}}
\]
2.7 Center of Mass

$N$ mass elements. If $r_j$ is the position of the $j$th element, and $m_j$ is its mass, then center of mass is defined as $\vec{R} = \frac{1}{M} \sum_{j=1}^{N} m_j \vec{r}_j$

2.7.1 Center of Mass of Continuous System

The result is not rigorous, since the mass elements are not true particles. However, in the limit where $N$ approaches infinity, the size of each element approaches zero and the approximation becomes exact.

$\vec{R} = \lim_{N \to \infty} \frac{1}{M} \sum_{j=1}^{N} m_j \vec{r}_j$. This limiting process defines an integral. Formally

$$\lim_{N \to \infty} \sum_{j=1}^{N} m_j r_j = \int \vec{r} \, dm$$

where $dm$ is a differential mass element. Then

$$\vec{R} = \frac{1}{M} \int \vec{r} \, dm$$

To visualize this integral, think of $dm$ as the mass in an element of volume $dV$ located at position $r$. If the mass density at the element is $\rho$, then $dm = \rho \, dV$ and $\vec{R} = \frac{1}{M} \int \vec{r} \, \rho \, dV$. This integral is called a volume integral.

Example: A rod of length $L$ has a non-uniform density $\lambda$, the mass per Unit length of the rod, varies as $\lambda = \lambda_0 (s / L)$ where $\lambda_0$ is a constant and $s$ is the distance from the end marked 0. Find the center of mass.

Solution: It is apparent that $\vec{R}$ lies on the rod. Let the origin of the coordinate system coincide with the end of the rod, 0, and let the $x$ axis lie along the rod so that $s = x$. The mass in an element of length $dx$ is $dm = \lambda_0 x \, dx / L$. The rod extends from $x = 0$ to $x = L$ and the total mass is $M = \int_0^L \lambda_0 x \, dx / L = \frac{1}{2} \lambda_0 L$.

The center of mass is at $\vec{R} = \frac{1}{M} \int r \, \lambda \, dM = \frac{2}{\lambda_0 L} \int_0^L \left( x \hat{i} + 0 \hat{j} + 0 \hat{k} \right) \lambda_0 x \, dx / L = \frac{2}{L^2} \left[ \frac{x^3}{3} \right]_0^L = \frac{2}{3} L \hat{i}$.
**Example:** Find the center of mass of a thin rectangular plate with sides of length $a$ and $b$, whose mass per unit area $\sigma$ varies in the following fashion:

$$\sigma = \sigma_o \left( \frac{xy}{ab} \right),$$

where $\sigma_o$ is a constant.

**Solution:**

$$\bar{R} = \frac{1}{M} \iint \left( x\hat{i} + y\hat{j} \right) \sigma \, dx \, dy,$$

We find $M$, the mass of the plate, as follows:

$$M = \int_0^b \int_0^a \sigma \, dx \, dy = \int_0^b \int_0^a \sigma_o \frac{xy}{ab} \, dx \, dy$$

We first integrate over $x$, treating $y$ as a constant.

$$M = \int_0^b \left( \int_0^a \sigma_o \frac{xy}{ab} \, dx \right) \, dy = \int_0^b \left( \sigma_o \frac{y}{b} \left[ \frac{x^2}{2a} \right]_0^a \right) \, dy = \int_0^b \sigma_o \frac{y}{b} \frac{a^2}{2a} \, dy = \frac{\sigma_o a^2}{2} \frac{b}{2} \bigg|_{y=0}^{y=b} = \frac{1}{4} \sigma_o ab.$$

The $x$ component of $R$ is

$$X = \frac{1}{M} \iint x \, dx \, dy = \frac{1}{M} \int_0^b \left( \int_0^a x \sigma_o \frac{xy}{ab} \, dx \right) \, dy = \frac{1}{M} \int_0^b \left( \sigma_o \frac{y}{b} \left[ \frac{x^3}{3a} \right]_0^a \right) \, dy = \frac{1}{M} \int_0^b \sigma_o \frac{y}{b} \frac{a^3}{3a} \, dy = \frac{1}{M} \sigma_o \frac{a^3 b^2}{3} = \frac{4 \sigma_o a^2 b}{6} = \frac{2 a}{3}.$$

Similarly, $Y = \frac{2}{3} b$.

So center of mass is $\bar{R} = \frac{2}{3} (a\hat{i} + b\hat{j})$.

**Example:** Find the center of mass of a uniform solid hemisphere of radius $R$ and mass $M$. From symmetry it is apparent that the center of mass lies on the $z$ axis, as illustrated. Its height above the equatorial plane is

$$Z = \frac{1}{M} \int z \, dM.$$
Solution: The integral is over three dimensions, but the symmetry of the situations lets us treat it as a one dimensional integral. We mentally subdivide the hemisphere into a pile of thin disks. Consider the circular disk of radius $r$ and thickness $dz$. Its volume is $dV = \pi r^2 dz$, and its mass is $dM = \rho dV = \left(\frac{M}{V}\right)(dV)$, where $V = \frac{2}{3}\pi R^3$. Hence,

$$Z = \frac{1}{M} \int M \frac{dz}{V} = \frac{1}{V} \int_{z=0}^{R} \pi r^2 z \, dz$$

To evaluate the integral we need to find $r$ in terms of $z$. Since $r^2 = R^2 - z^2$, we have

$$Z = \frac{\pi}{V} \int_{0}^{R} z \left(R^2 - z^2 \right) \, dz = \frac{\pi}{V} \left( \frac{1}{2} z^2 R^2 - \frac{1}{4} z^4 \right)_{0}^{R}$$

$$= \frac{\pi}{V} \left( \frac{1}{2} R^4 - \frac{1}{4} R^4 \right) = \frac{1}{4} \frac{\pi R^4}{\frac{2}{3} \pi R^3} = \frac{3}{8} R.$$

2.8 Conservation of Momentum

In the last section we found that the total external force $F$ acting on a system is related to the total momentum $P$ of the system by $F = \frac{dP}{dt}$.

Consider the implications of this for an isolated system, that is, a system which does not interact with its surroundings. In this case $F = 0$, and $dP/dt = 0$. The total momentum is constant; no matter how strong the interactions among an isolated system of particles, and no matter how complicated the motions, the total momentum of an isolated system is constant. This is the law of conservation of momentum. As we shall show, this apparently simple law can provide powerful insights into complicated systems.
2.8.1 Elastic Collision in One Dimension

Consider two elastic bodies $A$ and $B$ moving along the same line (figure). The body $A$ has a mass $m_1$ and moves with a velocity $v_1$ towards right and the body $B$ has a mass $m_2$ and moves with a velocity $v_2$ in the same direction. We assume $v_1 > v_2$ so that the two bodies may collide.

Let $v'_1$ and $v'_2$ be the final velocities of the bodies after the collision. The total linear momentum of the two bodies remains constant, so that,

$$m_1v_1 + m_2v_2 = m_1v'_1 + m_2v'_2 \quad \text{(i)}$$

or

$$m_1v_1 - m_1v'_1 = m_2v'_2 - m_2v_2 \quad \text{(ii)}$$

Also, since the collision is elastic, the kinetic energy before the collision is equal to the kinetic energy after the collision. Hence,

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v'_1^2 + \frac{1}{2}m_2v'_2^2$$

or

$$m_1v_1^2 - m_1v'_1^2 = m_2v'_2^2 - m_2v_2^2 \quad \text{(iii)}$$

or,

$$m_1(v_1^2 - v'_1^2) = m_2(v'_2^2 - v_2^2) \quad \text{(iii)}$$

Dividing (iii) by (ii)

$$v_1 + v'_1 = v'_2 + v_2$$

$$v_1 - v'_2 = v'_2 - v'_1 \quad \text{(iv)}$$

Now, $(v_1 - v_2)$ is the rate at which the separation between the bodies decreases before the collision. Similarly, $(v'_2 - v'_1)$ is the rate of increase of separation after the collision. So the equation (iv) may be written as

Velocity of separation (after collision) = Velocity of approach (before collision)...
This result is very useful in solving problems involving elastic collision. The final velocities $v_1'$ and $v_2'$ may be obtained from equation (i) and (iv). Multiply equation (iv) by $m_2$ and subtract from equation (i).

$$2m_2v_2 + (m_1 + m_2)v_1'$$

or

$$v_1' = \frac{(m_1 - m_2)}{m_1 + m_2}v_1 + \frac{2m_2}{m_1 + m_2}v_2$$

(1)

Now multiply equation (iv) by $m_1$ and add to equation (i),

$$2m_1v_1 - (m_1 - m_2)v_2 = (m_2 + m_1)v_2'$$

or

$$v_2' = \frac{2m_1v_1}{m_1 + m_2} - \frac{(m_1 - m_2)v_2}{m_1 + m_2}$$

(2)

Equations, (1) and (2) give the final velocities in terms of the initial velocities and the masses.

**Special Cases:**

(a) Elastic collision between a heavy body and a light body:

Let $m_1 >> m_2$. A heavy body hits a light body from behind.

We have,

$$\frac{m_1 - m_2}{m_1 + m_2} \approx 1, \quad \frac{2m_2}{m_1 + m_2} \approx 0 \quad \text{and} \quad \frac{2m_1}{m_1 + m_2} \approx 2$$

With these approximations the final velocities of the bodies are, from (1) and (2),

$$v_2' \approx v_1 \quad \text{and} \quad v_2' \approx 2v_1 - v_2$$

The heavier body continues to move with almost the same velocity. If the lighter body were kept at rest $v_2 = 0, v_2' = 2v_1$, which means the lighter body, after getting a push from the heavier body will fly away with a velocity double the velocity of the heavier body.

Next suppose $m_2 >> m_1$. A light body hits a heavy body from behind.

We have,

$$\frac{m_1 - m_2}{m_1 + m_2} \approx -1, \quad \frac{2m_2}{m_1 + m_2} \approx 2 \quad \text{and} \quad \frac{2m_1}{m_1 + m_2} \approx 0$$

The final velocities of the bodies are, from (1) and (2),

$$v_1' \approx -v_1 + 2v_2 \quad \text{and} \quad v_2' \approx v_2$$
The heavier body continues to move with almost the same velocity, the velocity of the lighter body changes. If the heavier body were at rest, \( v_2 = 0 \) then \( v'_1 = -v_1 \) the lighter body returns after collision with almost the same speed. This is the case when a ball collide elastically with a fixed wall and returns with the same speed.

(b) Elastic collision of two bodies of equal mass:

Putting \( m_1 = m_2 \) in equation (1) and (2) \( v'_1 = v_2 \) and \( v'_2 = v_1 \).

When two bodies of equal mass collide elastically, their velocities are mutually interchanged.

2.8.2 Perfectly Inelastic Collision in One Dimension

Final Velocity

When perfectly inelastic bodies moving along the same line collide, they stick to each other. Let \( m_1 \) and \( m_2 \) be the masses, \( v_1 \) and \( v_2 \) be their velocities before the collision and \( V \) be the common velocity of the bodies after the collision. By the conservation of linear momentum,

\[
m_1v_1 + m_2v_2 = m_1V + m_2V
\]

or

\[
V = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}
\]  

(i)

2.8.3 Elastic Collision in Two Dimensions

Consider two objects \( A \) and \( B \) of mass \( m_1 \) and \( m_2 \) kept on the \( X \)-axis (figure). Initially, the object \( B \) is at rest and \( A \) moves towards \( B \) with a speed \( u_1 \). If the collision is not head-on (the force during the collision is not along the initial velocity), the objects move along different lines. Suppose the object \( A \) moves with a velocity \( \vec{v} \) making an angle \( \theta \) with the \( X \)-axis and the object \( B \) moves with a velocity \( \vec{v}_2 \) making an angle \( \phi \) with the same axis. Also, suppose \( \vec{v}_1 \) and \( \vec{v}_2 \) lie in \( X - Y \) plane. Using conservation of momentum in \( X \) and \( Y \) directions, we get

\[
m_1u_1 = m_1v_1 \cos \theta + m_2v_2 \cos \theta
\]

and

\[
0 = m_1v_1 \sin \theta - m_2v_2 \sin \phi
\]

If the collision is elastic, the final kinetic energy is equal to the initial kinetic energy. Thus,

\[
\frac{1}{2} m_1u_1^2 = \frac{1}{2} m_1v_1^2 + \frac{1}{2} m_2v_2^2
\]  

(iii)
We have four unknowns $v_1, v_2, \theta$, and $\phi$ to describe the final motion whereas there are only three relations. Thus, the final motion cannot be uniquely determined with this information.

Example: A loaded spring gun, initially at rest on a horizontal frictionless surface, fires a marble at angle of elevation $\theta$. The mass of the gun is $M$, the mass of the marble is $m$, and the muzzle velocity of the marble is $v_0$. What is the final motion of the gun?

Solution: Take the physical system to be the gun and marble, Gravity and the normal force of the table act on the system. Both these forces are vertical. Since there are no horizontal external forces the $x$ component of the vector equation $F = dP/dt$ is

$$0 = \frac{dP_x}{dt} \quad (1)$$

According to equation (1) $P_x$ is conserved:

$$P_{x, \text{initial}} = P_{x, \text{final}} \quad (2)$$
Let the initial time be prior to firing the gun. Then $P_{x,\text{initial}} = 0$. Since the system is initially at rest.

After the marble has left the muzzle, the gun recoils with some speed $V_f$, and its final horizontal momentum is $MV_f$, to the left. Finding the final velocity of the marble involves a subtle point, however. Physically, the marble’s acceleration is due to the force of the gun, and the gun’s recoil is due to the reaction force of the marble. The gun stops accelerating once the marble leaves the barrel, so that at the instant the marble and the gun part company, the gun has its final speed $V_f$.

At that same instant the speed of the marble relative to the gun is $v_0$. Hence, the final horizontal speed of the marble relative to the table is $v_0 \cos \theta - V_f$. By conservation of horizontal momentum we therefore have

$$0 = m(v_0 \cos \theta) - V_f - MV_f$$

or

$$V_f = \frac{mv_0 \cos \theta}{M + m}$$

**Example:** Two identical buggies 1 and 2 with one man in each move without friction due to inertia along the parallel rails toward each other. When the buggies get opposite each other, the men exchange their places by jumping in the direction perpendicular to the motion direction. As a consequence, buggy 1 stops and buggy 2 keeps moving in the same direction, with its velocity becoming equal to $v$. Find the initial velocities of the buggies $v_1$ and $v_2$ if the mass of each buggy (without a man) equals $M$ and the mass of each man $m$.

**Solution:** status of buggies just before jump

**status of buggies just during jump**
status of buggies after jump

\[
\begin{align*}
\text{Buggy (1)} & \quad \text{Buggy (2)} \\
\begin{array}{c}
\text{\(m\)} \\
\text{\(M\)} \\
\text{\(v = 0\)}
\end{array} & \quad \begin{array}{c}
\text{\(\text{\(M\)}} \\
\text{\(V\)} \\
\text{\(m\)}
\end{array}
\end{align*}
\]

During this exchange momentum will be conserved because there is no force is horizontal direction.

Conservation of momentum for buggy (2) \( MV_1 - mv_2 = 0 \) \( \ldots(i) \)

Conservation of momentum for buggy (2) \( MV_2 - mV_1 = (m + M)V \) \( \ldots(ii) \)

From (i) and (ii):

\[
\begin{align*}
M M V V & = = - \quad \ldots(i) \\
& = = - \quad \ldots(ii)
\end{align*}
\]

But in term of vector: \( V_2 \) has opposite direction as \( V_1 \).

Then \( V_2 = \frac{MV}{M-m} \) and \( V_1 = \frac{-mV}{M-m} \)

2.9 Variable Mass and Momentum

Case 1-Motion of Spacecraft.

A spacecraft moves through space with constant velocity \( v \). The spacecraft encounters a stream of dust particles which embed themselves in it at rate \( \frac{dm}{dt} \). The dust has velocity \( u \) just before it hits. At time \( t \) the total mass of the spacecraft is \( M(t) \). The problem is to find external force \( F \) necessary to keep the spacecraft moving uniformly. (In practice \( F \) would most likely come from the spacecraft’s own rocket engines. For simplicity, we can visualize the source \( F \) to be completely external – an invisible hand, so to speak.)

Let us focus on the short time interval between \( t \) and \( t + \Delta t \). The drawings below show the system at the beginning and end of the interval.
Let $\Delta m$ denote the mass added to the satellite during $\Delta t$. The system consists of $M(t)$ and $\Delta m$. The initial momentum is $P(t)=M(t)v+(\Delta m)u$.

The final momentum is $P(t+\Delta t)-P(t)=(v-u)\Delta m$.

The rate of change of momentum is approximately

$$\frac{\Delta P}{\Delta t}=(v-u)\frac{\Delta m}{\Delta t}.$$  

In the limit $\Delta t \to 0$, we have the exact result

$$\frac{dP}{dt}=(v-u)\frac{dm}{dt}.$$  

Since $F=dP/dt$, the required external force is $F=(v-u)\frac{dm}{dt}$.

Note that $F$ can be either positive or negative depending on the direction of the stream of mass.

If $u=v$, the momentum of the system is constant, and $F=0$.

**Example:** Sand falls from a stationary hopper onto a freight car which is moving with uniform velocity $v$. The sand falls at the rate $dm/dt$. How much forces needed to keep the freight car moving a the speed $v$?
Solution: In this case, the initial speed of the sand is 0, and
\[
\frac{dP}{dt} = (v-u)\left(\frac{dm}{dt}\right) = v\frac{dm}{dt}
\]
The required force is \( F = v\frac{dm}{dt} \). We can understand why this force is needed by considering in detail just what happens to a sand grain as it lands on the surface of the freight car.

Case -2 Motion of Rocket

The concept of momentum is invaluable in understanding the motion of a rocket. A rocket accelerates by expelling gas at a high velocity; the reaction force of the gas on the rocket accelerates the rocket in the opposite direction. The mechanism is illustrated by the drawings of the cubical chamber containing gas at high pressure.

The gas presses outward on each wall with the force \( F_a \) (We show only four walls for clarity.) The vector sum of the \( F_a \)'s is zero, giving zero net force on the chamber. Similarly each wall of the chamber exerts a force on the gas \( F_b = -F_a \); the net force on the gas is also zero. In the right hand drawings below, one wall has been removed. The net force on the chamber is \( F_a \), to the right. The net force on the gas is \( F_b \), to the left. Hence the gas accelerates to the left, and the chamber accelerates to the right.
To analyze the motion of the rocket in detail, we must equate the external force on the system, $F$, with the rate of change of momentum, $dP/dt$. Consider the rocket at time $t$. Between $t$ and $t + \Delta t$ a mass of fuel $\Delta m$ is burned and expelled as gas with velocity $u$ relative to the rocket. The exhaust velocity $u$ is determined by the nature of the propellants, the throttling of the engine, etc., but it is independent of the velocity of the rocket.

The sketches below show the system at time $t$ and at time $t + \Delta t$. The system consists of $\Delta m$ plus the remaining mass of the rocket $M$. Hence the total mass is $M + \Delta m$.

The velocity of the rocket at time $t$ is $v(t)$, and at $t + \Delta t$ it is $v + \Delta v$.

The initial momentum is $P(t) = (M + \Delta m)$

the final momentum is $P(t + \Delta t) = M(v + \Delta v) + \Delta m(v + \Delta v + u)$

The change in momentum is $\Delta P = P(t + \Delta t) - P(t) = M\Delta v + (\Delta m)u$ assume $\Delta m, \Delta v = 0$

Therefore, 

$$\frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = M\frac{dv}{dt} + u\frac{dm}{dt}.$$ 

Note that we have defined $u$ to be positive in the direction of $v$. In most rocket applications, $u$ is negative, opposite to $v$. It is inconvenient to have both $m$ and $M$ in the equation. $dm/dt$ is the rate of increase of the exhaust mass. Since this mass comes from the rocket,

$$\frac{dm}{dt} = -\frac{dM}{dt}.$$ 

equating the external force to $dP/dt$, we obtain the fundamental rocket equation

$$F = M\frac{dv}{dt} - u\frac{dM}{dt}. $$
It may be useful to point out two minor subtleties in our development. The first is that the velocities have been expressed with respect to an inertial frame, not a frame attached to the rocket. The second is that we took the final velocity of the element exhaust gas to be $v + \Delta v + u$ rather than $v + u$.

**Example:** (a) A rocket is moving vertically in absence of gravity. If initial exhaust speed and mass of rocket is $u$ and $M_0$, respectively. What will be speed if mass became $\alpha M_0$ after time $t = t_f$. (assume $t = 0$ velocity of rocket is $v_0$.)

(b) How situation will different if a gravitational field is present.

**Solution:** (a) If there is no external force on a rocket, $F = 0$ and its motion is given by

$$M \frac{dv}{dt} = u \frac{dM}{dt} \quad \text{or} \quad \frac{dv}{dt} = \frac{u}{M} \frac{dM}{dt}$$

Generally the exhaust velocity $i$ is constant, in which case it is easy to integrate the equation of motion.

$$\int_{v_0}^{v_f} dv = \int_0^{M_0} \frac{dM}{M} \quad \text{or} \quad v_f - v_0 = u \ln \frac{M_f}{M_0} = u \ln \alpha$$

(b) If a rocket takes off in a constant gravitational field,

$$M \frac{dv}{dt} = -u \frac{dM}{dt} + g$$

Integrating with respect to time, we obtain

$$v_f - v_0 = u \ln \left( \frac{M_f}{M_0} \right) + g \left(t_f - t_0\right) \Rightarrow v_f = v_0 + u \ln (\alpha) + gt_f.$$
MCQ (Multiple Choice Questions)

Q1. Match the following List I with List II.

<table>
<thead>
<tr>
<th>List I</th>
<th>List II</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Elastic collision</td>
<td>1. Momentum constant but K.E. is not constant</td>
</tr>
<tr>
<td>B. Inelastic collision</td>
<td>2. Moment and K.E. is constant.</td>
</tr>
<tr>
<td>C. Perfectly inelastic collision</td>
<td>3. $0 &lt; e &lt; 1$</td>
</tr>
<tr>
<td>D. Partly elastic collision</td>
<td>4. $e = 0$</td>
</tr>
</tbody>
</table>

A B C D
(a) 2 4 3 1
(b) 2 1 4 3
(c) 1 2 3 4
(d) 4 3 1 2

Q2. Two bodies of mass 100 g and 20 g are moving with velocities $\left(2\hat{i} - 7\hat{j} + 3\hat{k}\right)$ and $\left(-10\hat{i} + 35\hat{j} + 3\hat{k}\right)$ cm/s respectively. The velocity of the centre of mass is given by.

(a) $2k m/s$
(b) $\left(-8\hat{i} + 28\hat{j}\right) cm/s$
(c) $\left(\frac{10}{2}\hat{i} - \frac{35}{6}\hat{j} + 3\hat{k}\right) cm/s$
(d) $2k cm/s$

Q3. A 2 kg mass is acted upon by a force which gives a time dependent displacement of $x = \frac{t^2}{2}$. The work done in the time interval 0 to $t_0$ is

(a) $0.5t_0$
(b) $t_0$
(c) $t_0^2$
(d) $t_0^3$

Q4. A body of mass $m$, moving with velocity $u$ collides elastically with another body at rest having mass $M$. If the body of mass $M$ moves with velocity $v$, then the velocity of the body of mass $m$ after the impact is

(a) $\frac{m-M}{m+M}u$
(b) $\frac{m-M}{m+M}v$
(c) $\frac{mu + Mv}{m + M}$
(d) $\frac{mu - Mv}{m + M}$
Q5. The mass of a rocket is $M$ and the total mass of rocket and the fuel is $M_0$. The average exhaust velocity of gases ejected from rocket motors is $u$ are the final velocity attained by the rocket after using all fuel is $v$. The final velocity $v$ is proportional to

(a) $\log\left(\frac{M_0}{M}\right)$  
(b) $\left(\frac{Mu}{M_0}\right)$  
(c) $(M_0 - M)$  
(d) $(M_0 - M)^{-1}$

Q6. A block of mass 2 kg is attached with a spring of spring constant 4900 Nm$^{-1}$ and the system is kept on smooth, horizontal plane the other end of spring is attached with a wall initially, spring is stretched by 5 cm from its natural length and the block is at rest. Now, suddenly an impulse of 4 kg ms$^{-1}$ is given to the block towards the wall. The velocity of block, when spring acquires natural length is

(a) 5 ms$^{-1}$  
(b) 3 ms$^{-1}$  
(c) 6 ms$^{-1}$  
(d) None

Q7. Pick out the **WRONG** alternative.
(a) A force is said to be conservative if the work done by the force, when a particle moves in a closed loop, is zero.
(b) If the work done by a force is path independent then the force is said to be conservative.
(c) Work done by frictional force is path independent.
(d) Frictional force is a non-conservative.

Q8. Pick out the correct alternative.
(a) The concept of potential energy can be associated with any type of force.
(b) The mechanical energy of a system is the sum of kinetic potential and internal energy.
(c) The total mechanical energy of a system remains constant if the internal forces are conservative and external forces do no work.
(d) Suppose the internal forces within a system are conservative and external forces do negative work then the mechanical energy of the system increases.
Q9. Pick out the WRONG alternative:
(a) When an object of mass \(m\) moves from height \(h_1\) above the earth surface to height \(h_2\) the change in its gravitational potential energy is \(mg(h_2 - h_1)\).
(b) A block of mass \(m\) slides down a frictionless incline, then the speed of block of the bottom is \(\sqrt{2gh}\) where \(h\) is the height of incline.
(c) When the free end of a spring is displaced from \(x_1\) to \(x_2\), the change in its potential energy is \(\frac{1}{2}k(x_2^2 - x_1^2)\), where \(k\) is the spring constant.
(d) The change in potential energy of a system is equal to the work done by the conservative force on the system.

Q10. A particle is rotated in a vertical circle by connecting it to a spring of length \(l\) and keeping to other end of the string fixed. The minimum speed of the particle when the string is horizontal for which the particle will complete the circle is
(a) \(\sqrt{gl}\)   (b) \(\sqrt{2gl}\)   (c) \(\sqrt{3gl}\)   (d) \(\sqrt{5gl}\)

Q11. Two springs \(A\) and \(B(\frac{k_A}{k_B} = 2)\) are stretched by applying forces of equal magnitude at the four ends. If the energy stored in \(A\) is \(E\), that in \(B\) is
(a) \(\frac{E}{2}\)   (b) \(2E\)   (c) \(E\)   (d) \(\frac{E}{4}\)

Q12. Consider the following two statements
(A) Linear momentum of the system remains constant.
(B) Centre of mass of the system remains at rest.
(a) A implies B and B implies A
(b) A does not imply B and B does not imply A
(c) A implies B and B does not imply A
(d) B implies A but A does not imply B
Q13. A body falling vertically downwards under gravity breaks in two parts of unequal masses. The centre of mass of the two parts taken together shifts horizontally towards
(a) heavier piece  
(b) lighter piece  
(c) does not shift horizontally  
(d) depends on the vertical velocity of the time of breaking

Q14. A body of rest breaks into two pieces of equal masses. The parts will move
(a) in the same direction  
(b) along different lines  
(c) in opposite directions with equal speeds  
(d) in opposite directions with unequal speeds

Q15. Pick out the correct alternative
(a) If the net external force acting on a system is zero, its acceleration is non-zero.  
(b) If the velocity of centre of mass is constant, then linear momentum is necessarily constant.  
(c) During collision of a system of particles linear momentum is not constant.  
(d) Conservation of momentum for a system implies the conservation of energy for that system.

Q16. Two balls are thrown simultaneously in air. The acceleration of the centre of mass of the two balls while in air
(a) depends on the direction of motion of the balls  
(b) depends on the masses of two balls  
(c) depends on the speed of the two balls  
(d) is equal to $g$

Q17. The potential energy function associated with the force $\vec{F} = 4xy\hat{i} + 2x^2\hat{j}$ is
(a) $u = -2x^2y$  
(b) $u = -2xy^2 + \text{constant}$  
(c) $u = 2x^2y + \text{constant}$  
(d) Not defined
Q18. When a spring is stretched by 2 cm, it stores 100 J of energy. If it is stretched further by 2 cm, the stored energy will be increased by
(a) 100 J  (b) 200 J  (c) 300 J  (d) 400 J

Q19. The bob of a pendulum of length 2 m lies at P. When it reaches Q, it loses 10% of its total energy due to air resistance. The velocity at Q is (Take $g = 10 \text{ m/s}^2$).
(a) 6 m/s  (b) 1 m/s  (c) 2 m/s  (d) 8 m/s

Q20. A block moving horizontally on a smooth surface with a speed of 20 m/s breaks into two equal parts continuing in the same direction. If one of the parts moves at 30 m/s, with what speed does the second part move.
(a) 5 m/s  (b) 10 m/s  (c) 15 m/s  (d) 20 m/s

Q21. A car of mass $M$ is moving with a uniform velocity $v$ on a horizontal road when a man drops himself on it from above. Taking the mass of the man to be $m$, the velocity of the car after the event is
(a) $\frac{3mv}{M+m}$  (b) $\frac{mv}{M+m}$  (c) $\frac{Mv}{M+m}$  (d) $\frac{(M+m)v}{M}$

Q22. You lift a suitcase from the floor and keep it on a table. The work done by you on the suitcase does not depend on
(a) the path taken by the suitcase  (b) the time taken by you in doing so
(c) the weight of the suitcase  (d) your weight
NAT (Numerical Answer Type)

Q23. 2 kg mass is tied to one end of a string 2 m in length which is hanging vertically. What minimum horizontal speed should be imparted to the mass such that it just reaches the top of the vertical circle with equal to the length of the string? (Take \( g = 10 \text{ m/s}^2 \)) _______ m/s

Q24. A 10^4 kg rocket standing on its launch pad can expel exhaust gas at the speed of 2 × 10^3 m/s. Just to lift the rocket from the ground, the exhaust gas must be ejected at ____ kg s^{-1}

Q25. A 25 kg mass, starting from rest at the top, slides down a plane that makes an angle of 30° with the horizontal. When it reaches the bottom of the 10 m long slide, its velocity is 8 m/s. The work done by the force of friction is closest to a value of ______ J

Q26. An object of mass \( m \) moving with a velocity \( v \) approaching a second object of the same mass but at rest. The total kinetic energy of the two objects viewed from the centre of mass is ……….. \( amv^2 \) the value of \( \alpha \)

Q27. Work done when a force \( \vec{F} = (\hat{x} + 2\hat{y} + 3\hat{z}) \) acting on a particle takes it from the point \( \vec{r}_1 = (\hat{x} + \hat{y} + \hat{z})m \) to the point \( \vec{r}_2 = (\hat{x} - \hat{y} + 2\hat{z})m \) is ______ J

Q28. A mass of 0.5 kg moving with a speed of 1.5 m/s on a horizontal smooth surface, collides with a weightless spring of force constant \( k = 50 \text{ N/m} \). The maximum compression of the spring would be ______ m.

Q29. A moving body of mass \( m \) and velocity 3 m/s collides with a motionless body of mass 2m and sticks to it. Now the velocity of the combined mass would be _______ m/s.

Q30. A block of mass \( M \) is attached to the lower end of a vertical spring. The spring is hung from the ceiling and has force constant value \( k \). The mass is released from rest with the spring initially unstretched. The maximum extension produced in the length of the spring will be _______ \( Mg/k \).
Q31. A particle is projected at an angle of 60° with the horizontal with a kinetic energy $E$. The kinetic energy of the highest point is $\alpha E$, then value of $\alpha$ is _______.

Q32. A uniform chain of length 2 m is kept on a table such that a length of 60 cm hangs freely from the edge of the table. The total mass of the chain is 4 kg. The work done in pulling the entire chain on the table is ________ J.

Q33. A particle of mass $4m$ explodes into three fragments. Two of the fragments each of mass $m$ are found to move with a speed is each in mutually perpendicular directions. The total energy released in the process of explosion is ________ $mv^2$.

Q34. Two particles of mass 1 kg and 3 kg move towards each other under their mutual force of attraction. No other force acts on them. When the relative velocity of approach of two particles is $2 m/s$, their centre of mass has a velocity of $0.5 m/s$. When the relative velocity of approach becomes $3 m/s$, the velocity of centre of mass is ________.

Q35. A particle of mass 10 kg is moving with velocity of $(2\hat{i} + 3\hat{j}) m/s$. From time $t = 0$, a force $10 \hat{j}$ starts acting on the particle. The $x$- component of momentum of the particle at any other time $t$ is ________.

Q36. Two blocks of masses 10 kg and 4 kg are connected by a spring of negligible mass and placed on a frictionless horizontal surface. An impulse gives a velocity of 14 $m/s$ to the heavier block in the direction of the lighter block. The velocity of centre of mass is ______.

Q37. A 4.0 kg block sliding on a frictionless surface explodes into two fragments of 2.0 kg parts. One moves with 3.0 $m/s$ due north and other at 6.0 $m/s$, 30° north east. The original speed of the block is ____________.

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MSQ (Multiple Select Questions)

Q38. If for a system of $N$ particles of different masses $m_1, m_2, ..., m_N$ with position vectors $\vec{r}_1, \vec{r}_2, ..., \vec{r}_N$ and corresponding velocities $\vec{v}_1, \vec{v}_2, ..., \vec{v}_N$ respectively such that $\sum \vec{v}_i = 0$, then

(a) linear momentum must be constant
(b) total angular momentum must be independent of the choice of the origin
(c) the total force on the system must be zero
(d) total torque on the system must be zero

Q39. A ball bits the floor and rebounds, after elastic collision in this case

(a) the magnitude of the momentum of the ball just after collision is same as that just before collision
(b) the mechanical energy of the ball remains the same in the collision.
(c) total momentum of ball and earth is conserved
(d) total kinetic energy of ball and earth is conserved

Q40. A net force $F_x(t) = A + Bt^2$ in the $+x$ direction is applied to a boy of mass $m$. The force starts at $t = 0$ and continues until time $t$.

(a) Impulse of the force is $2Bt$
(b) His speed at time $t$ is $\frac{A}{m}t + \frac{B}{3m}t^3$
(c) Impulse of the force is $At + \frac{B}{3}t^3$
(d) Speed at time $t$ is $\frac{2B}{m}t$

Q41. A block of mass $m$ moving on a smooth horizontal plane with a velocity $v_0$ collides with a stationary block of mass $M$ at the back of which a spring of spring constant $k$ is attached as shown in figure. Select the correct option(s).

(a) Velocity of centre of mass $\frac{m}{m+M}v_0$
(b) Initial kinetic energy of the system in centre of mass frame is $\frac{1}{4}\left(\frac{mM}{m+M}\right)v_0^2$
(c) Maximum compression in the spring is $v_0\sqrt{\frac{mM}{(M+m)}\frac{1}{k}}$
(d) When the spring is in state of maximum compression the kinetic energy in the centre of mass frame is zero.
Q42. A body moving towards a finite body at rest collides with it. It is possible that
(a) both bodies come to rest
(b) both bodies move after collision
(c) the moving body came to rest and stationary body starts moving
(d) the stationary body remain stationary the moving body change its velocity

Q43. Pick out the CORRECT alternative(s):
(a) If center of mass of three particles is at rest and it is known that two of them are moving along different lines, then the third must also be moving.
(b) If centre of mass remains at rest, then net work done by the forces acting on the system must be zero.
(c) If centre of mass remains at rest, then the net external force must be zero.
(d) For two particles collision, the linear momentum is conserved but for three particles it is not conserved.

Q44. A block of 4 kg mass starts at rest and slides a distance \( d \) down a friction less incline (angle 30°) when it runs into a spring of negligible mass. The block slides an additional difference of 25 cm before it is brought to rest momentarily by compressing the spring. The force constant of the spring is 400 N/m.
(i) The value of \( d \) is
(ii) When the block strikes the spring its speed is \( \sqrt{3.75} \) m/s
(iii) At maximum compression the potential energy of the spring is 12.5 J
(iv) The change in gravitational P.E. of the block from the top and till the position of momentary rest is 15 J

Q45. A body moving towards a body of finite mass at rest collides with it. It is possible that
(a) both bodies come to rest.
(b) both bodies move after collision.
(c) the moving body stops and the body at rest starts moving.
(d) the stationary body remains stationary.
Q46. In an elastic collision between two particles.
   (a) the total kinetic energy of the system is always conserved.
   (b) the kinetic energy of the system before collision is equal to the kinetic energy of the system after collision.
   (c) the linear momentum of the system is always conserved.
   (d) the linear momentum before collision is equal to the linear momentum after collision.

Q47. A particle is taken from point $A$ to point $B$ under the influence of a force field. Now it is taken from $B$ to $A$ and it is observed that the work done in taking the particle from $B$ to $A$ is not equal to the work done in taking it from $A$ to $B$. If $W_{nc}$ and $W_c$ are the work done by non-conservative and conservative forces and $\Delta U$ and $\Delta K$ are change in potential and change in kinetic energy then
   (a) $W_{nc} - \Delta U = \Delta K$  
   (b) $W_{nc} = -\Delta U$  
   (c) $W_{nc} + W_c = \Delta K$  
   (d) $W_{nc} - \Delta U = -\Delta K$

Q48. In the adjoining figure block $A$ is of mass $m$ and block $B$ is of mass $2m$. The spring has force constant $K$. All the surfaces are smooth and the system is released from rest with spring unstretched.
   (a) The maximum extension of the spring is $\frac{4mg}{K}$
   (b) The speed of block $A$ when extension in spring is $\frac{2mg}{K}$, is $2g \sqrt{\frac{2m}{3K}}$
   (c) The net acceleration of block $B$ when the extension in the spring is maximum is $\frac{2}{3}g$
   (d) Tension in the thread for extension of $\frac{2mg}{K}$ in spring is $mg$.

Q49. At two positions kinetic energy and potential energy of a particle are $K_1 = 10J$, $U_1 = -20J$, $K_2 = 20J$, $U_2 = -10J$. In moving from 1 to 2.
   (a) Work done by conservative forces is positive
   (b) Work done by conservative forces is negative
   (c) Work done by all the forces is positive
   (d) Work done by all the forces is negative
Q50. The potential energy function of a particle in the \(x-y\) plane is given by \(U = k(x+y)\) where \(k\) is a constant

(a) The work done by the conservative force in moving a particle from \((1,1)\) to \((2,3)\) is \(-3k\)
(b) The work done by the conservative force in moving the particle from \((0,0)\) to \((1,1)\) is \(-2k\)
(c) The work done by the conservative force in moving the particle from \((1,1)\) to \((2,2)\) is \(+2k\)
(d) The work done by the conservative force in moving the particle from \((0,0)\) to \((3,3)\) is \(-6k\)

Q51. A block is placed on top of an inclined plane inclined at \(37^\circ\) with the horizontal. The length of the plane is 5 m. The block slides down the plane and reaches the bottom. Take \(g=10\,\text{m/s}^2\), \(\sin 37^\circ = \frac{3}{5}\) and \(\cos 37^\circ = \frac{4}{5}\).

(a) If the plane is smooth the block reaches the bottom with a speed of \(2\sqrt{15}\,\text{m/s}\)
(b) If the plane is smooth the block reaches the bottom with a speed of \(3\sqrt{15}\,\text{m/s}\)
(c) If the plane has the coefficient of friction 0.25, the block reaches the bottom with a speed of \(2\sqrt{10}\,\text{m/s}\)
(d) If the plane has coefficient of friction 0.25, the block reaches the bottom with a speed of \(5\sqrt{10}\,\text{m/s}\)

Q52. You lift a suitcase from the floor and keep it on a table. The work done by you on the suitcase does not depend on

(a) the path taken by the suitcase   (b) the time taken by you in doing so
(c) the weight of the suitcase    (d) your weight
MCQ (Multiple Choice Questions)

Ans. 1: (b)  
Ans. 2: (d)

Solution: If \( \mathbf{r}_1, \mathbf{r}_2 \) are position vectors of two masses \( m_1 \) and \( m_2 \) then the position vector of their centre of mass \( \mathbf{R}_{CM} \) is given as \( (m_1 + m_2) \mathbf{R}_{CM} = m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 \)

Differentiating w.r.t. time, we get \( (m_1 + m_2) \mathbf{v}_{CM} = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 \)

\[
100 \mathbf{v}_{CM} = 100 \left(2\hat{i} - 7\hat{j} + 3\hat{k}\right) + 20 \left(-10\hat{i} + 35\hat{j} - 3\hat{k}\right)
\]

\[
120 \mathbf{v}_{CM} = 200\hat{i} - 700\hat{j} + 300\hat{k} + (-200\hat{i}) + 700\hat{j} - 60\hat{k} = 240\hat{k}
\]

\[
\mathbf{v}_{CM} = \frac{240}{120} \mathbf{k} = 2\hat{k} \text{ cm/s}
\]

Ans. 3: (c)

Solution: The displacement is given as \( x = \frac{t^2}{2} \Rightarrow dx = \frac{2}{2} t \, dt \) Work done \( \Rightarrow dW = F \, dx = madx \)

(ii) Differentiating equation (i) w.r.t. time, we get

Acceleration \( \frac{d^2 x}{dt^2} = \frac{2}{2} = 1 \) So force \( F = ma = 2 \times 1 \Rightarrow F = 2N \)

Work done \( dW = 2dx = 2t \, dt \Rightarrow W = \int_0^t t \, dt = 2 \cdot \frac{t_0^2}{2} = t_0^2 \)

Ans. 4: (a)

Solution: When the collide elastically then their velocities after collision are given as

\[
v_1 = \frac{(m_1 - m_2)u_1 + 2m_2u_2}{m_1 + m_2}
\]

\[
v_2 = \frac{2m_1u_1 + (m_2 - m_1)u_2}{m_1 + m_2}
\]

Here, \( m_1 = m, m_2 = M, u_1 = u, u_2 = 0 \)

And \( v_2 = v \)

Before collision  \( \rightarrow \)  After collision
Putting all these values in equation (i), we get

\[ v_i = \frac{(m-M)u + 0}{m+M} \Rightarrow v_i = \frac{(m-M)}{m+M} u \]

Ans. 5: (a)

Solution: The velocity of rocket

\[ M_0 = \text{mass at } t=0 \quad v_0 = \text{velocity at } t=0 \quad v = v_0 + v_e \log \frac{M_0}{M}, v \propto \log \frac{M_0}{M} \]

Ans. 6: (b)

Solution: Applying conservation of energy

\[ \frac{1}{2} kx^2 + \frac{m^2 v^2}{2m} = \frac{1}{2} mv_0^2 + 0 \Rightarrow mv^2 = kx^2 + \frac{m^2 v^2}{m} \]

\[ \Rightarrow 2xv^2 = 4000 \times \left( 5 \times 10^{-2} \right)^2 + \frac{4}{2} \Rightarrow 2v^2 = 0.4 \times 25 + 8 \Rightarrow v' = 9 \Rightarrow v' = 3 \]

Ans. 7: (c)

Solution: A force is said to be conservative if the work done by it in a closed loop is zero. Alternatively if the work done by a force is path independent, the force is said to be conservative. Frictional force does not satisfy any of these properties, hence it is a non-conservative force.

Ans. 8: (c)

Solution: The concept of potential energy can be associated with only a conservative force. The mechanical energy of a system is the sum of kinetic and potential energy. Internal conservative forces can not change the mechanical energy. The mechanical energy of the system changes by the amount of work done by the external force.

Ans. 9: (d)

Solution: Gravitational potential energy at a height \( h \) is \( mgh \), hence

\[ \Delta U = mg \left( h_2 - h_1 \right) \]

Mechanical energy of position 1

\[ = \text{Mechanical energy of position 2, hence } 0 + mgh = \frac{1}{2} mv^2 + 0 \]

\[ \therefore v = \sqrt{2gh} \]

The change in spring potential energy is

\[ \Delta U = \frac{1}{2} K \left( x^2_2 - x^2_1 \right) \]

The change in potential energy of a system is equal to the negative of the work done by the conservative force on the system.
Ans. 10: (c)

Solution: The minimum speed at the top for the particle to complete the circle is \( \sqrt{gl} \). Now applying the principle of conservation of mechanical energy.

Mechanical energy of position 1 = Mechanical energy of position 2

\[
\frac{1}{2}mv^2 + mgl = \frac{1}{2}m\left(\sqrt{gl}\right)^2 + 2mgl
\]

or by solving we get \( v = \sqrt{3gl} \)

Ans. 11: (b)

Solution: \( F = k_Ax_A = k_Bx_B \) or \( \frac{x_B}{x_A} = \frac{k_A}{k_B} = 2 \)

Hence

\[
\frac{E_B}{E_A} = \frac{1}{2}k_Bx_B^2 = \left(\frac{1}{2}\right) \cdot 2^2 = 2 \quad \therefore E_B = 2E_A = 2E
\]

Ans. 12: (d)

Solution: Linear momentum of a system is \( \vec{P} = MV_{cm} \)

If \( \vec{P} \) is constant we can only conclude that \( V_{cm} \) is constant, if \( V_{cm} \) is constant then \( \vec{P} \) is constant.

Ans. 13: (d)

Solution: Since the motion of centre of mass is decided by the net external force acting on the system and the net external force acting on the system and the net external force is the same (both before and after breaking), hence centre of mass does not shift at all.

Ans. 14: (c)

Solution: From conservation of momentum

\[
M \times 0 = \frac{M}{2} \vec{v}_1 + \frac{M}{2} \vec{v}_2 \Rightarrow \vec{v}_1 = -\vec{v}_2
\]

Ans. 15: (b)

Solution: From the relation \( \vec{F}_{ext} = M\vec{a}_{cm} \) if \( \vec{F}_{ext} = 0 \), then \( \vec{a}_{cm} = 0 \)

From the relation \( \vec{P} = MV_{cm} \), if \( V_{cm} \) is constant, \( \vec{P} \) is constant. During collision the linear momentum is necessarily constant (in the absence of external forces).

Conservation of linear momentum and conservation of energy are entirely different concepts.
Ans. 16: (d)
Solution: From the relation $\vec{F}_{\text{ext}} = m\ddot{a}_c$ we have $mg = m\ddot{a}_c$ or $\ddot{a}_c = \ddot{g}$

Ans. 17: (b)
Solution: From the relation $\vec{F} = -\frac{\partial U}{\partial x} \hat{i} - \frac{\partial U}{\partial y} \hat{j}$

We see that the most general expression is $U = -2x^2y + \text{constant}$.

Ans. 18: (c)
Solution: From the question,
\[
\frac{1}{2} k (2 \text{cm})^2 = 100 \quad \text{(i)}
\]
\[
\frac{1}{2} k (4 \text{cm})^2 = E \quad \text{(ii)}
\]
Hence $\frac{E}{100} = 4 \Rightarrow E = 400 J$, change in energy $= 400 - 100 = 300 J$.

Ans. 19: (a)
Solution: Work done by air resistance = decrease in mechanical energy
\[
\Rightarrow 0.1 mgl = \frac{1}{2} m v^2 - mgl \quad \therefore v = \sqrt{1.8 gl} = \sqrt{36} = 6 \text{ m/s}
\]

Ans. 20: (b)
Solution: $M \cdot 20 = \frac{M}{2} \cdot 30 + \frac{M}{2} v$ \quad or \quad $40 = 30 + v \Rightarrow v = 10 \text{ m/s}$

Ans. 21: (b)
Solution: Even after the vertical fall, the horizontal component of momentum is not affected, hence
\[
Mv = (M + m)v' \quad \text{or} \quad v' = \frac{Mv}{M + m}
\]

Ans. 22: (c)
Solution: since the suitcase is at rest in both situation, hence
\[
W_g + W_{\text{you}} = 0 \quad \text{(work KE theory)}
\]
\[
W_{\text{you}} = -W_g = -(mg \cdot h) = mgh \quad \text{. Hence only option (c) is correct}
\]
NAT (Numerical Answer Type)

Ans. 23: 10 m/s

Solution: If \( u \) is velocity at the lowest point \( A \) then in order en reach the mass at highest point \( B \) the velocity \( u \) should be given as \( u^2 \geq 5gr \).

\[
u = \sqrt{5gr} = \sqrt{5 \times 10 \times 2} = \sqrt{100} = 10 \text{ m/s}
\]

Ans. 24: 49 kg s\(^{-1}\)

Solution: If \( \frac{dm}{dt} \) is the rate exhaust gas then it is related with speed of rocket as

\[
v \frac{dm}{dt} = Mg \Rightarrow 2 \times 10^3 \frac{dm}{dt} = 10^4 \times 9.8 \Rightarrow \frac{dm}{dt} = \frac{9.8 \times 10^4}{2 \times 10^3} = 49 \text{ kg s}^{-1}
\]

Ans. 25: 450

Solution: By conservation of energy law.

Total energy at \( A \) = Total energy at \( O \)

\[
\Rightarrow (KE)_A + (PE)_A = (KE)_O + (PE)_O + \text{(Frictional energy)}
\]

\[
0 + MgAB = \frac{1}{2} mv^2 + 0 + W_i \Rightarrow Mg 10 \sin 30 = \frac{1}{2} mv^2 + W_i
\]

\[
\Rightarrow 10 \times 25 \times 10 \times \frac{1}{2} = \frac{1}{2} \times 25 \times (8)^2 + W_i \Rightarrow 1250 = 800 + W_i \Rightarrow W_i = 450 \text{ J}
\]

Ans. 26: 0.25

Solution: Let \( v_1 \) is velocity of first mass after collision and \( v_2 \) is velocity of second mass after collision. By the conservation law momentum

\[
m_1v = m_1v_1 + m_2v_2
\]

But \( m_1 = m_2 = m \Rightarrow mv = mv_1 + mv_2 \Rightarrow v = v_1 + v_2 \)

By conservation law of energy

\[
\frac{1}{2} mv^2 = \frac{1}{2} mv_1^2 + \frac{1}{2} mv_2^2 \Rightarrow v^2 = v_1^2 + v_2^2
\]

Solving equation (ii) and (iii), we get

Velocity of mass becomes zero whereas velocity of sector mass attain the velocity of first.
Now, the velocity of centre of mass is given

\[(m + m)\text{v}_{CM} = m\text{v} \Rightarrow \text{v}_{CM} = \frac{1}{2}\text{v}\]

Hence, the velocity of first w.r.t. centre of mass

\[0 - \text{v}_{CM} = -\frac{1}{2}\text{v}\]

Total kinetic energy as seen centre of mass

\[
m(v_1 - v_{cm})^2 + m(v_2 - v_{cm})^2 = m\left(\frac{v_1 - \text{v}}{2}\right)^2 + m\left(\frac{v_1 - \text{v}}{2}\right)^2
\]

using \(v^2 = v_1^2 + v_2^2\) and \(v = v_1 + v_2\) = \(E = \frac{mv^2}{4}\)

Ans. 27: \(-1\)

Solution: Work done = \(\vec{F}.d\vec{r}\)

Here, \(\vec{F} = \hat{x} + 2\hat{y} + 3\hat{z}\ N\)

\(\vec{r}_1 = (\hat{x} + \hat{y} + \hat{z}); \vec{r}_2 = (\hat{x} - \hat{y} + 2\hat{z}), dr = (2\hat{y} + \hat{z})\)

So, \(F = (\hat{x} + 2\hat{y} + 3\hat{z}).(-2\hat{y} + \hat{z}) = 2(-2) + 3(1) = -4 + 3 = -1 J\)

Ans. 28: \(0.15\)

Solution: Taking block & spring as the system and applying the conservation of mechanical energy

\[
\frac{1}{2} \times 0.5 \times (1.5)^2 + 0 = 0 + \frac{1}{2} \times 50 \times x_{max}^2 \Rightarrow x_{max} = 0.15 m
\]

Ans. 29: \(1\)

Solution: From conservation of linear momentum \(m \times 3 = (m + 2m) \times v\) or \(v = \frac{3m}{3m} = 1 m/s\)

Ans. 30: \(2\)

Solution: Taking the zero of spring potential energy of the location of unstretched spring and the zero of gravitational potential energy at the lowest point.

From conservation of mechanical energy \(0 + Mg \frac{d^2}{2} + 0\) or \(d = \frac{2Mg}{k}\)
Ans. 31: 0.25

Solution: The height attained by the projectile

\[ H = \frac{v_0^2 \sin^2 \theta_0}{2g} \], where \( v_0 \) is the speed of projection and \( \theta_0 \) is the initial angle with the horizontal.

\[ \frac{v_0^2}{2g} \cdot \frac{3}{4} \quad \text{or} \quad H = \frac{3v_0^2}{8g} \]

From conservation of mechanical energy \( E + 0 = \frac{mg \cdot 3v_0^2}{8g} \)

But \( \frac{1}{2}mv^2 = E \quad \Rightarrow E = K + \frac{3}{4}E \Rightarrow K = \frac{1}{4}E \)

Ans. 32: 3.6

Solution: Work done in putting the chain = change in gravitational potential energy

\[ = 0 - \left[ -\left(\frac{4}{2}\right)(0.6) \times 10 \times (0.3) \right] = 3.6 \text{ J} \]

Ans. 33: 1.5

Solution: From conservation of momentum

\[ 0 = mv' + mv' + 2mv' \quad \text{or} \quad v' = -\frac{v}{2} i - \frac{v}{2} j \]

\[ \frac{1}{2} \cdot 2mv^2 = \frac{1}{2} \cdot 2m \left( \frac{v^2}{4} + \frac{v^2}{4} \right) = \frac{1}{2}mv^2 \]

hence total energy released

\[ \frac{1}{2}mv^2 + \frac{1}{2}mv^2 + \frac{1}{2}mv^2 = \frac{3}{2}mv^2 = 1.5mv^2 \]

Ans. 34: 0.5 m/s

Solution: When no net external force acts on a system the total linear momentum and hence the velocity of centre of mass remains constant. Hence in both cases velocity of centre of mass 0.5 m/s.
Ans. 35: \(20 \text{ kg m/s}^{-1}\)

Solution: At \(t = 0\), the \(x\)-component of momentum is \(10 \times 2 = 20 \text{ kg m/s}^{-1}\)

Since no net external force acts on the particle in the \(x\)-direction, hence its \(x\)-component of momentum remains constant.

Ans. 36: \(10 \text{ m/s}\)

Solution: The velocity of centre of mass is given by

\[
V_{cm} = \frac{m_1\dot{v}_1 + m_2\dot{v}_2}{m_1 + m_2}
\]

\[
\therefore V_{cm} = \frac{10 \times 14 + 4.0}{10 + 4} = 10 \text{ m/s}
\]

Ans. 37: \(3.43 \text{ m/s}\)

Solution: From conservation of momentum

\[
M\ddot{V} = m_1\ddot{v}_1 + m_2\ddot{v}_2
\]

or

\[
4\ddot{V} = 2\left(6\cos30\hat{i} + 6\sin30\hat{j}\right) + 2 \times 3\hat{j}
\]

\[
4\ddot{V} = 2\left(\frac{6\sqrt{3}}{2}\hat{i} + \frac{6}{2}\hat{j}\right) + 6\hat{j} = 6\sqrt{3}\hat{i} + 3\hat{j} + 3\hat{j} = 6\sqrt{3}\hat{i} + 9\hat{j}
\]

\[
\ddot{V} = \frac{6\sqrt{3}}{4}\hat{i} + \frac{9}{4}\hat{j} \Rightarrow V = \frac{1}{4}\sqrt{108 + 81} = \frac{\sqrt{189}}{4} \text{ m/s} = 3.43 \text{ m/s}
\]

MSQ (Multiple Select Questions)

Ans. 38: (a) and (c)

Solution: \(\ddot{v}_{cm} = \frac{m_1\ddot{v}_1 + m_2\ddot{v}_2 + \ldots + m_N\ddot{v}_N}{m_1 + m_2 + \ldots + m_N} \Rightarrow \ddot{v}_{cm} = \sum_{i=1}^{N} \frac{m_i\ddot{v}_i}{\sum m_i} \Rightarrow \sum \ddot{v}_i = 0
\]

\[
\Rightarrow \sum \frac{d\ddot{v}_i}{dt} = 0 \Rightarrow \frac{1}{m_i} \sum m_i \frac{d\ddot{v}_i}{dt} = 0 \Rightarrow \frac{1}{m_i} \sum dF_i = 0 \Rightarrow \sum dF_i = 0
\]

Hence, total force on the system is zero so linear momentum is conserve.
Ans. 39: (c) and (d)
Ans. 40: (b) and (c)

Solution: \( F = A + Bt^2 \)

Impulse, \( I = \int_0^t F \, dt = \int_0^t A + Bt^2 = At + \frac{Bt^3}{3} \)

Acceleration, \( a = \frac{F}{m} = \frac{A + Bt^2}{m} \Rightarrow \frac{dv}{dt} = \frac{A + Bt^2}{m} \Rightarrow \int_0^v dv = \int_0^t \frac{A + Bt^2}{m} \, dt \Rightarrow v = \frac{A}{m} t + \frac{B}{3m} t^3 \)

Ans. 41: (a), (c) and (d)

Solution: Velocity of centre of mass \( v_{cm} = \frac{m v_0 + M \times 0}{m + M} \Rightarrow v_{cm} = \frac{m v_0}{m + M} \)

For maximum compression, velocity of each mass become same let it is \( V \)

\[ \Rightarrow \frac{1}{2} m v_0^2 = \frac{1}{2} (m + M) V^2 \Rightarrow V = \sqrt{\frac{m}{m + M}} v_0 \]

\[ \Rightarrow \frac{1}{2} k x^2 = \frac{1}{2} m v_0^2 - \frac{1}{2} \left( \frac{m^2}{m + M} \right) v_0^2 - \frac{1}{2} \left( \frac{m}{m + M} \right) v_0^2 \]

on solving, \( x = u_0 \sqrt{\frac{m M}{M + m} \frac{1}{k}} \)

Initial KE \( = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} M v_{cm}^2 = \frac{1}{2} \times \frac{m^2}{m + M} v_0^2 [m + M] = \frac{1}{2} \left( m + m \right) v_0^2 \)

Ans. 42: (b), (c) and d)

Ans. 43: (a) and (c)

Solution: By definition \( \vec{v}_{cm} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3}{m_1 + m_2 + m_3} \)

\( \vec{v}_{cm} = 0 \) and \( \vec{v}_1 \neq 0 \) and \( \vec{v}_2 \neq 0 \) and the lines of \( \vec{v}_1 \) and \( \vec{v}_2 \) are different implies \( m_1 \vec{v}_1 + m_2 \vec{v}_2 \neq 0 \)

hence \( \vec{v}_3 = \frac{- (m_1 \vec{v}_1 + m_2 \vec{v}_2)}{m_3} \neq 0 \)
If centre of mass is at rest then \( \ddot{a}_{cm} = 0 \) hence \( \vec{F}_{\text{exp}} = 0 \), but this does not imply net work done by the forces acting on the system to be zero. Linear momentum (in the absence of external force) is conserved for collision of any number of particles.

Ans. 44: (a), (b) and (c)

Solution: Loss of gravitational potential energy = gain in spring potential energy

\[
mg \sin \theta (d + 0.25) = \frac{1}{2} kd^2 \Rightarrow 4 \times 10 \times \frac{1}{2} (d + 0.25) = \frac{1}{2} \times 400 \times (0.25)^2
\]

\( d + 0.25 = 0.375, d = 0.375 \text{ m} \quad \text{or} \quad d = 37.5 \text{ cm} \)

The speed of the block at the time of striking the spring is

\[
mgd \sin 30^0 = \frac{1}{2} mv^2 \quad \Rightarrow 10 \times 0.375 \times \frac{1}{2} = \frac{1}{2} v^2 \quad \Rightarrow v = \sqrt{3.75} \text{ m/s}
\]

At maximum compression potential energy of spring is

\[
U = \frac{1}{2} kx^2 \quad \text{or} \quad U = \frac{1}{2} \times 400 \times (0.25)^2 = 12.5 \text{ J}
\]

Ans. 45: (b), (c) and (d)

Solution: Since one body is moving the net linear momentum of the system is non-zero. If both bodies come to rest the net linear momentum will be zero which is not possible.

Yes both bodies may move so as to conserve momentum.

If this is an elastic collision of two equal masses, then the moving body will stop and the body at rest will move.

If the stationary body is of infinite mass, it will remain at rest and the moving body will rebound.

Ans. 46: (b), (c) and (d)

Solution: In elastic collision, the kinetic energy is not conserved during the process of collision, but it is conserved just before and just after the collision.

In any type of collision (in the absence of external forces) the momentum is conserved.

Ans. 47: (a) and (c)

Solution: By definition \( W_c = -\Delta U \) also \( W_c + W_{nc} = \Delta k \)

Hence, \(-\Delta U + W_{nc} = \Delta k\)
Ans. 48: (a)

Solution: Initially the system is at rest and finally the system is at rest. Applying the principle of conservation of mechanical energy for maximum compression.

Loss in gravitational potential energy = given in spring potential energy

\[ 2mgx_{\text{max}} = \frac{1}{2} kx_{\text{max}}^2 \quad \text{or} \quad x_{\text{max}} = \frac{4mg}{k} \]

when the extension is \( \frac{2mg}{k} \), applying it conservation of mechanical energy gives

\[ 2m \cdot \frac{2mg}{k} g = \frac{1}{2} mv^2 + \frac{1}{2} \cdot 2mv^2 \Rightarrow \frac{4m^2 g^2}{k} = \frac{3mv^2}{2} \quad \text{or} \quad v = \sqrt{\frac{8mg}{3k}} = 2 \sqrt{\frac{2mg}{3k}} \]

Ans. 49: (b) and (c)

Solution: \( W_C = -\Delta U \), hence \( W_C = -(-10 + 20) = -10 \ J \)

From work-kinetic energy theorem

Work done by all forces = charge in kinetic energy

\[ \Rightarrow \quad W_{\text{all}} = 20 - 10 = 10 \ J \]

Ans. 50: (a), (b) and (d)

Solution: Work done by conservation force = \( -\Delta U \)

\[ W_{(1,1)\rightarrow(2,3)} = -3k \quad W_{(0,0)\rightarrow(1,1)} = -2k \quad W_{(1,1)\rightarrow(2,2)} = -2k \quad W_{(0,0)\rightarrow(3,3)} = -6k \]

Ans. 51: (a) and (c)

Solution: If the plank is smooth, applying the principle of conservation of mechanical energy

\[ 5mg \sin 37^0 = \frac{1}{2} mv^2 \quad \text{or} \quad 5 \times 10 \times \frac{3}{5} \times \frac{1}{2} \times v^2 \Rightarrow v^2 = \sqrt{60} = 2 \sqrt{15} \ m/s \]

If the plank has friction, work done by friction force = change in mechanical energy

\[ \Rightarrow \quad ( -\mu mg \cos 37^0 ) (5) = \left( \frac{1}{2} mv^2 + 0 \right) - \left( 0 + 5mg \sin 37^0 \right) \]

\[ \Rightarrow \quad -5 \times (0.25) \times 10 \times \frac{4}{5} = \frac{v^2}{2} - 5 \times 10 \times \frac{3}{5} \Rightarrow -10 = \frac{v^2}{2} - 30 \]

\[ \frac{v^2}{2} = 20.0 \quad : \quad v^2 = 40 \Rightarrow v = 2 \sqrt{10} \ m/s \]
Ans. 52: (a), (b) and (d)

Solution: since the suitcase is at rest in both situation, hence

\[ W_g + W_{you} = 0 \] (work KE theory)

\[ W_{you} = -W_g = -(mg)h = mgh \]
3. Central Force and Kepler’s System

3.1 Central Force

In classical mechanics, the central-force problem is to determine the motion of a particle under the influence of a single central force. A central force is a force that points from the particle directly towards (or directly away from) a fixed point in space, the center, and whose magnitude only depends on the distance of the object to the center.

In central force potential $V$ is only function of $r$ only a central force is always a conservative force; the magnitude $F$ of a central force can always be expressed as the derivative of a time-independent potential energy

$$V = \frac{1}{r} F(r) \sin(\theta)
\frac{\partial}{\partial r} \left( \frac{\partial F}{\partial \phi} \right) \frac{\partial}{\partial \theta} = 0$$

And the force $F$ is defined as $F = -\frac{\partial V}{\partial r} \hat{r}$ (force is only in radial direction)

3.1.1 Angular Momentum and Areal Velocity

The equation of motion in polar coordinate is given by $m(\ddot{r} - r \dot{\theta}^2) = F_r$ and $m(\dot{r} \dot{\theta} + 2 \ddot{r}) = F_\theta$

but for central force

$$\tau = \vec{r} \times \vec{F}, \quad \Rightarrow \tau = r \hat{r} \times \frac{\partial V}{\partial r} \hat{r}$$

External torque $\tau = 0$ so angular momentum is conserve

$m(r \dot{\theta} + 2 \dot{r}) = F_\theta$ but for central force $F_\theta = 0$ so $m(r \dot{\theta} + 2 \dot{r}) = 0 \Rightarrow \frac{d(mr^2 \dot{\theta})}{dt} = 0$ means

Angular momentum of $mr^2 \dot{\theta} = J = |\vec{r} \times \vec{p}| \quad \Rightarrow \dot{\theta} = \frac{J}{mr^2}$

$\vec{r} \cdot \vec{J} = \vec{r} \cdot (\vec{r} \times \vec{p}) = 0 \quad \Rightarrow \vec{r} \perp \vec{J}$ hence position vector $\vec{r}$ is perpendicular to angular momentum vector $\vec{J}$ hence $\vec{J}$ is conserve , its magnitude and direction both are fixed so direction of $\vec{r}$ is also fixed.

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So motion due to central force is confined into a plane and angular momentum \( \mathbf{j} \) is perpendicular to that plane.

Central force problem. Prove that Areal velocity is constant.

For the central force problem. Now \( A = \frac{1}{2} r \cdot rd \theta \)

Areal velocity \( \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d \theta}{dt} = \frac{1}{2} r^2 \dot{\theta} \)

\( \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} \) It is given that \( \dot{\theta} = \frac{J}{m r^2} \) so \( \frac{dA}{dt} = \frac{J}{2m} \)

Which means equal area will swept in equal time

3.1.2 Total Energy of the System

Hence total energy is not explicitly function of time \( t \) so \( \frac{\partial E}{\partial t} = 0 \) one can conclude that total energy in central potential is constant.

\[ E = \frac{1}{2} m v^2 + V(r) \text{ and Velocity } \mathbf{v} = \dot{r} \mathbf{r} + r \dot{\theta} \mathbf{j} \]

So total energy \( E = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + V(r) \)

\[ = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + V(r) \text{ it is known } \dot{\theta} = \frac{J}{m r^2} \]

\[ = \frac{1}{2} m \dot{r}^2 + \frac{J^2}{2m r^2} + V(r) \text{ \( r > 0 \)} \]

\( \Rightarrow E = \frac{1}{2} m \dot{r}^2 + V_{\text{eff}} \) Where \( \frac{J^2}{2m r^2} + V(r) \) is identified as effective potential \( V_{\text{effective}} \)
3.1.3 Condition for Circular Orbit

From equation of motion in radial part \( m(\ddot{r} - r\dot{\theta}^2) = f(r) \)

For circular orbit or radius \( r_0 \), \( r = r_0 \) and \( \dot{r} = 0 \) \(-r\dot{\theta}^2 = f(r) \Rightarrow -\frac{J^2}{mr^3} = f(r) \) at \( r = r_0 \)

Which can be also derived by \( \frac{\partial V_{\text{effective}}}{\partial r} \bigg|_{r=r_0} = 0 \) and \( \dot{\theta} = \omega_0 \) is identified as angular frequency in circular orbit.

radius \( r = r_0 \) of circular orbit is also identified as stable equilibrium point so \( \frac{\partial^2 V_{\text{effective}}}{\partial r^2} \bigg|_{r=r_0} \geq 0 \). if some how particle of mass \( m \) is change its orbit without changing is angular momentum and new orbit is bounded then new orbit is identified as elliptical orbit. the angular frequency in new elliptical orbit is \( \omega = \sqrt{\frac{\partial^2 V_{\text{effective}}}{\partial r^2} \bigg|_{r=r_0}} \)

3.1.4 Equation of Motion and Differential Equation of Orbit

From equation of motion in radial part \( m(\ddot{r} - r\dot{\theta}^2) = f(r) \Rightarrow \frac{m d^2 r}{dt^2} - \frac{J^2}{mr^3} = f(r) \) \( \ldots \ldots \ldots (1) \)

Where \( J = mr^2 \dot{\theta} \Rightarrow d\theta = \frac{J}{mr^2} dt \Rightarrow \frac{d}{dt} = \frac{J}{mr^2} \frac{d}{d\theta} \)

\( \frac{d^2}{dt^2} = \left( \frac{d}{dt} \right) \left( \frac{d}{dt} \right) = \left( \frac{J}{mr^2} \right) \frac{d}{d\theta} \left( \frac{J}{mr^2} \right) \frac{d}{d\theta} \)

Substitute in (1)

\( \frac{J^2}{m} \frac{1}{r^2} \frac{d}{d\theta} \left( \frac{1}{r} \frac{dr}{d\theta} \right) - \frac{J^2}{mr^3} = f(r) \Rightarrow \frac{J^2}{m} \frac{1}{r^2} \frac{d}{d\theta} \left( \frac{d(-1/r)}{d\theta} \right) - \frac{J^2}{mr^3} = f(r) \)

\(- \left( \frac{J^2}{m} \frac{1}{r^2} \frac{d^2(1/r)}{d\theta^2} + \frac{J^2}{mr^3} \right) = f(r) \Rightarrow - \frac{J^2}{mr^2} \left( \frac{d^2(1/r)}{d\theta^2} \right) + \frac{1}{r} = f(r) \)

Let \( \frac{1}{r} = u \Rightarrow - \frac{J^2 u^2}{m} \left( \frac{d^2 u}{d\theta^2} + u \right) = f \left( \frac{1}{u} \right) \) (differential equation of an orbit)
Example: Consider that the motion of a particle of mass $m$ in the potential field $V(r) = \frac{kr^2}{2}$ if $l$ is angular momentum.

(a) What is effective potential ($V_{\text{eff}}$) of the system. plot $V_{\text{eff}}$ vs $r$

(b) Find value of energy such that motion is circular in nature.

(c) If particle is slightly disturbed from circular orbit such that its angular remain constant. What will nature of new orbit? Find the angular frequency of new orbit in term of $m, l, k$.

Solution: (a) $V_{\text{eff}} = \frac{J^2}{2mr^2} + \frac{1}{2} kr^2$

(b) $\frac{dV_{\text{eff}}}{dr} = -\frac{J^2}{mr^3} + kr = 0$ at $r = r_0$ so $r_0 = \left(\frac{J^2}{mk}\right)\frac{1}{4}$ and $J = m\omega_0 r_0^2$

for circular motion $m\omega_0^2 r_0 = kr_0$ where $r_0$ is radius of circle $\omega_0 = \sqrt{\frac{k}{m}}$

total energy $E = \frac{J^2}{2mr^2} + \frac{1}{2} kr^2 = \frac{mk^4}{2mr_0^2} + \frac{1}{2} kr_0^2$ $E = kr_0^2$ put $r_0 = \left(\frac{J^2}{mk}\right)^{1/4}$ $E = J\sqrt{\frac{k}{m}}$

(c) orbit is elliptical in nature

$$\frac{d^2V_{\text{eff}}}{dr^2} \bigg|_{r=r_0} = \frac{3J^2}{mr^4} + k = \frac{3J^2}{m\left(\frac{J^2}{mk}\right)} + k = 4k$$

$$\omega = \sqrt{\frac{4k}{m}} \Rightarrow \omega = 2\sqrt{\frac{k}{m}} \Rightarrow 2\omega_0$$
Example: A particle of mass \( m \) moves under the influence of an attractive central force \( f(r) \).

(a) What is condition that orbit is circular in nature if \( J \) is the angular momentum of particle

(b) If force is in form of \( f(r) = \frac{-k}{r^n} \) determine the maximum value of \( n \) for which the circular orbit can be stable.

**Solution:** (a) if \( V_{\text{eff}} = \frac{J^2}{2mr^2} + V(r) \) for circular stable orbit \( \frac{\partial V_{\text{eff}}}{\partial r} = 0, \frac{\partial^2 V_{\text{eff}}}{\partial r^2} > 0 \)

(b) \( f(r) = \frac{-k}{r^n} \) for circular motion \( \frac{\partial V_{\text{eff}}}{\partial r} = 0 \Rightarrow \frac{J^2}{mr^2} + \frac{\partial V}{\partial r} = 0 \)

It is given \( \frac{\partial V}{\partial r} = -f(r) \) if \( f(r) = \frac{-k}{r^n} \Rightarrow \frac{\partial V}{\partial r} = \frac{k}{r^n} \)

\[-\frac{J^2}{mr^2} + \frac{k}{r^n} = 0 \Rightarrow \frac{k}{r^n} = \frac{J^2}{mr^2} \]

\[\frac{\partial^2 V_{\text{eff}}}{\partial r^2} > 0 \Rightarrow \frac{3J^2}{mr^4} - \frac{nk}{r^{n+1}} > 0 \Rightarrow \frac{3J^2}{mr^4} - \frac{n}{r} > 0 \Rightarrow \frac{J^2}{mr^3} > 0 \text{ so } n < 3\]

Example: A particle of mass \( m \) and angular momentum \( l \) is moving under the action of a central force \( f(r) \) along a circular path of radius \( a \) as shown in the figure. The force centre \( O \) lies on the orbit.

(a) Given the orbit equation in a central field motion.

\[\frac{d^2 u}{d\theta^2} + u = -\frac{m}{l^2 u^2} f, \text{ where } u = \frac{1}{r}.\]

Determine the form of force in terms of \( l, m, a \) and \( r \).

(b) Calculate the total energy of the particle assuming that the potential energy \( V(r) \to 0 \) as \( r \to \infty \).
Solution: (a) from the fig \( r = 2a \cos \theta \)
\[
\frac{1}{r} = \frac{\sec \theta}{2a}
\]
\[
-J^2u^2 \left[ \frac{d^2u}{d\theta^2} + u \right] = f\left(\frac{1}{u}\right)
\]
\[
-J^2 \sec^2 \theta \left[ \frac{1}{2a} \left( \sec \theta \tan^2 \theta + \sec^3 \theta \right) + \frac{\sec \theta}{2a} \right] = f\left(\frac{1}{u}\right)
\]
\[
-J^2 \sec^2 \theta \left[ \frac{1}{2a} \left( \sec \theta \tan^2 \theta + \sec^3 \theta + \sec \theta \right) \right] = f\left(\frac{1}{u}\right)
\]
\[
-J^2 \sec^3 \theta \left[ \tan^2 \theta + \sec^2 \theta + 1 \right] = f\left(\frac{1}{u}\right) \Rightarrow -\frac{2J^2 \sec^5 \theta}{4a^2 m} = f\left(\frac{1}{u}\right) \Rightarrow f(r) \propto \frac{1}{r^5}
\]

(b) \( E = \frac{mr^2}{2} + \frac{J^2}{2mr^2} + V(r) \), \( r \to \infty \), \( V(r) \to 0 \)
\[
E = \frac{mr^2}{2} \quad \text{and} \quad r = 2a \cos \theta \quad \text{and} \quad \dot{r} = -2a \sin \theta \dot{\theta}, \quad \dot{\theta} = \frac{J^2}{mr^2} \quad \text{as} \quad r \to \infty
\]

hence \( \dot{\theta} = \frac{J^2}{mr^2} \to 0 \quad \text{so} \quad \dot{r} \to 0 \quad \text{so} \quad E = 0 \)
3.2 Two Body Problem

Reduction of two body central force problem to the equivalent one body problem

A system of two particles of mass $m_1$ and $m_2$ whose instantaneous position vectors of inertial frame with origin $O$ are $r_1$ and $r_2$ respectively.

Vector $m_2$ relative to $m_1$ is $\mathbf{r} = r_2 - r_1$

The potential energy is $V$ is only function for distance between the particles

So $V = V(|\mathbf{r}_2 - \mathbf{r}_1|)$

Total energy of the system is given by in lab frame

$$E = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2 + V(r_2 - r_1)$$

Let the position vectors of $m_1$ and $m_2$ be $r_1$ and $r_2$. The position vector of the center of mass, measured from the same origin, is

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$

The center of mass lies on the line joining $m_1$ and $m_2$. To show this, suppose first that the tip of $\mathbf{R}$ does not lie on the line, and consider the vectors $\mathbf{r}'_1$, $\mathbf{r}'_2$ from the tip of $\mathbf{R}$ to $m_1$ and $m_2$. From the sketch we see that

$$\mathbf{r}'_1 = \mathbf{r}_1 - \mathbf{R}$$
$$\mathbf{r}'_2 = \mathbf{r}_2 - \mathbf{R}$$

$$\mathbf{r}'_1 = \mathbf{r}_1 - \frac{m_1 \mathbf{r}_1}{m_1 + m_2} - \frac{m_2 \mathbf{r}_2}{m_1 + m_2} = \frac{m_2}{m_1 + m_2} (\mathbf{r}_1 - \mathbf{r}_2)$$

$$\mathbf{r}'_2 = \mathbf{r}_2 - \frac{m_1 \mathbf{r}_1}{m_1 + m_2} - \frac{m_2 \mathbf{r}_2}{m_1 + m_2} = -\left( \frac{m_1}{m_1 + m_2} \right) (\mathbf{r}_1 - \mathbf{r}_2)$$

$\mathbf{r}'_1$ and $\mathbf{r}'_2$ are proportional to $r_1 - r_2$, the vector from $m_1$ to $m_2$. Hence $\mathbf{r}'_1$ and $\mathbf{r}'_2$ lie along the line joining $m_1$ and $m_2$ as shown. Furthermore,

$$\mathbf{r}'_1 = \frac{m_2}{m_1 + m_2} |r_1 - r_2| = \frac{m_2}{m_1 + m_2} r \quad \text{and} \quad \mathbf{r}'_2 = \frac{m_1}{m_1 + m_2} |r_1 - r_2| = \frac{m_1}{m_1 + m_2} r$$
The total energy is transformed

\[ E = \frac{1}{2} \left( m_1 + m_2 \right) \left[ \frac{\dot{r}_1^2}{r_1^2} + \frac{\dot{r}_2^2}{r_2^2} \right] + \frac{1}{2} m_1 m_2 \left[ \frac{\dot{r}_1^2}{r_1^2} + \frac{\dot{r}_2^2}{r_2^2} \right] - V(r) \]

Centre of mass moving with constant momentum and equation of motion for three generalized co-ordinates or will not terms in \( R \) and \( \dot{R} \). Hence discuss the motion of system one can ignore \( \frac{1}{2} (m_1 + m_2) \dot{R}^2 \).

So Energy in centre of mass reference frame reduce to

\[ E = \frac{1}{2} \left( m_1 m_2 \right) \left[ \frac{\dot{r}_1^2}{r_1^2} + \frac{\dot{r}_2^2}{r_2^2} \right] + V(r) \]

Where \( \mu = \frac{m_1 m_2}{m_1 + m_2} \) is reduce to one body system in centre of mass reference frame.

### 3.2.1 Kepler’s Problem

Kepler discuss the orbital motion of the sun and Earth system under the potential \( V(r) = -\frac{k}{r} \) where \( k = Gm_s m_e \) it is given \( m_s \) and \( m_e \) is mass of Sun and Earth. Although kepler discuss sun and earth system but method can be used for any system which is interacting with potential \( V(r) = -\frac{k}{r} \)

The reduce mass for sun and earth system is \( \mu = \frac{m_s m_e}{m_s + m_e} \Rightarrow \frac{m_e}{m_e + m_s} = m_e , m_s \gg m_e \)

Let us assume mass of earth \( m_e = m \)

### 3.2.2 Kepler’s First Law:

Every planet (earth) moves in an elliptical orbit around the sun, the sun is being at one of the foci. Where sun and earth interact each other with potential \( V(r) = -\frac{k}{r} \) we solve equation of motion in center of mass reference frame with reduce mass \( \mu = m_e = m \)
3.2.3 Equation of Motion:

\[
m\ddot{r} - m\dot{r}^2 = -\frac{k}{r^2}
\]

put \( \dot{\theta} = \frac{J}{mr^2} \)

\[
m\ddot{l} - \frac{l^2}{mr^3} = -\frac{k}{r^2}
\]

Equation or orbit is given by

\[
\frac{d^2u}{d\theta^2} \left[ u - \frac{km}{J^2} \right] = \frac{d^2u}{d\theta^2} + u = -\frac{1}{u}
\]

\[
f(r) = -\frac{k}{r^2} \Rightarrow f\left(\frac{1}{u}\right) = -ku^2 \Rightarrow \frac{d^2u}{d\theta^2} + u = +ku^2 \Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{ku^2m}{J^2u^2}
\]

\[
\frac{d^2u}{d\theta^2} \left[ u - \frac{km}{J^2} \right] = 0 \Rightarrow u = \frac{km}{J^2}
\]

The equation reduce to \( \frac{d^2y}{d\theta^2} + y = 0 \)

The solution of equation reduce to \( y = A\cos \theta \Rightarrow u = \frac{km}{J^2} + A\cos \theta \)

\[
\frac{1}{r} = \frac{km}{J^2} + A\cos \theta \Rightarrow \frac{J^2}{km} = 1 + \frac{AJ^2}{km} \cos \theta
\]

Put \( \frac{J^2}{km} = l \) and \( e = \frac{AJ^2}{km} \) the equation reduce to \( \frac{l}{r} = 1 + e\cos \theta \) which is equation of conics where \( l \) is latus rectum and \( e \) is eccentricity.

In a central force potential which is interacting with potential \( V(r) = -\frac{k}{r} \) can be any conics section depending on eccentricity \( e \).

Now we are discuss the case specially of elliptical orbit as Kepler discuss for planetary motion.

Total energy \( E = \frac{1}{2}mr^2 + \frac{J^2}{2mr^2} - \frac{k}{r} \) where \( V_{\text{effective}} = \frac{J^2}{2mr^2} - \frac{k}{r} \) with constant angular momentum \( J \).
If one will plot $V_{effective}$ Vs $r$ it is clear that for negative energy the orbit is elliptical which is shown in figure. Earth is orbiting in elliptical path with sun as focus as shown in figure.

Let equation of this ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $b = a\sqrt{1-e^2}$

minimum value of $r$ is $(a-ae)$ value and maximum value of $r$ is $(a+ae)$ $r_{max} + r_{min} = 2a$ from plot of effective potential it is identified $r_{max}$ and $r_{min}$ is the turning point so so at these points radial velocity is zero

$E = \frac{J^2}{2mr^2} - \frac{k}{r} \Rightarrow 2mEr^2 + 2mkr - J^2 = 0$ given equation is quadratic in term of $r$ for their root at $r_{max}$ and $r_{min}$. Using theory of quadratic equation sum of root $r_{max} + r_{min} = \frac{-2mk}{2mE}$

$\Rightarrow E = -\frac{k}{2a}$ which is negative.

3.2.4 Relationship between Energy and Eccentricity.

For central potential $V(r) = -\frac{k}{r}$ the solution of orbit is $l = \frac{1}{r} = 1 + e \cos \theta$ with $l = \frac{J^2}{km}$

The energy is given by $E = \frac{1}{2} mr^2 + \frac{J^2}{2mr^2} - \frac{k}{r}$

the solution of orbit is $l = 1 + e \cos \theta$ with $l = \frac{J^2}{km}$

so $\frac{-l}{r^2} \dot{r} = -e \sin \theta \dot{\theta}$ where $\dot{\theta} = \frac{J}{mr^2}$ so $\dot{r} = \frac{eJ \sin \theta}{ml}$,

after putting the value of $\frac{l}{r} = 1 + e \cos \theta$ and $\dot{r} = \frac{eJ \sin \theta}{ml}$ with $l = \frac{J^2}{km}$ in equation of energy,
one will get \( e = \sqrt{1 + \frac{2EJ^2}{mk^2}} \)

the condition on energy for possible nature of orbit for potential

\[
E > 0 \quad e > 1 \quad \text{Hyperbola} \\
E = 0 \quad e = 1 \quad \text{Parabola} \\
E < 0 \quad e < 1 \quad \text{Ellipse} \\
E = -\frac{mk^2}{2J^2} \quad e = 0 \quad \text{circle}
\]

3.2.5 Kepler’s Second Law:

Equal Area will swept in equal time or Areal velocity is constant.

\[
\frac{dA}{dt} = \frac{J}{2m} \quad \text{(which is derived earlier)}
\]

3.2.6 Kepler’s Third Law:

The square of time period \( T \) of revolution in elliptical orbit is proportional to cube of semi major axis \( a \) i.e \( T^2 \propto a^3 \)

\[
\frac{dA}{dt} = \frac{J}{2m} \Rightarrow \int dA = \frac{J}{2m} \int dt \Rightarrow \pi ab = \frac{J}{2m}T \quad (\pi ab \text{ is the area of ellipse})
\]

\[
\pi a \cdot a \sqrt{1 - e^2} = \frac{J}{2m}T \quad \text{it is given} \quad e = \sqrt{1 + \frac{2EJ^2}{mk^2}} \quad \text{and} \quad E = -\frac{k}{2a}
\]

\[
e^2 = 1 - \frac{2kJ^2}{2amk^2} \Rightarrow 1 - e^2 = \frac{2kJ^2}{2amk^2}
\]

\[
T^2 = \frac{4m^2}{J^2} \pi^2 a^2 \cdot a^2 (1 - e^2) \quad \text{put value of} \quad 1 - e^2 = \frac{2kJ^2}{2amk} \Rightarrow T^2 = \frac{4m^2}{J^2} \pi^2 a^4 \cdot \frac{J^2}{mak} = \frac{4\pi^2 ma^3}{k}
\]

\[
\Rightarrow T^2 = \frac{4\pi^2 ma^3}{k} \quad \text{if} \quad k = Gm_s m \quad \text{then} \quad T^2 = \frac{4\pi^2 a^3}{Gm_s} \quad \text{where} \quad m_s \text{ is mass of the sun.}
\]
Example: Given a classical model of tritium atom with atom with nucleus of charge +1 and a single.

Electron in a circular orbit of radius $r_0$ suddenly the nucleus emits a negatron and changes to charge +2 (the emitted negatron escapes rapidly and we can forget about it.) the orbit suddenly has a new situation.

(a) Find the ratio of the electron’s energy after to before the emission of the negatron

(b) Describe qualitatively the new orbit

(c) Find the distance of closest and the farthest approach for the new orbits in units of $r_0$

Solution: (a) As the negatron leaves the system rapidly, we can assume that its leaving has no effect on the position and kinetic energy of the orbiting electron.

From the force relation for the electron,

$$\frac{mv^2}{r_0} = \frac{e^2}{4\pi\varepsilon_0 r_0^2},$$

and we find its kinetic energy

$$\frac{mv^2}{2} = \frac{e^2}{8\pi\varepsilon_0 r_0}$$

and its total mechanical energy

$$E_1 = \frac{mv^2}{2} - \frac{e^2}{4\pi\varepsilon_0 r_0} = -\frac{e^2}{8\pi\varepsilon_0 r_0}$$

before the emission of the negatron. After the emission the kinetic energy of the electron is still $-\frac{e^2}{8\pi\varepsilon_0 r_0}$, while its potential energy suddenly changes to $\frac{-2e^2}{4\pi\varepsilon_0 r_0} = -\frac{e^2}{2\pi\varepsilon_0 r_0}$.

Thus after the emission the total mechanical energy of the orbiting electron is

$$E_2 = \frac{mv^2}{2} - \frac{2e^2}{4\pi\varepsilon_0 r_0} = \frac{-3e^2}{8\pi\varepsilon_0 r_0},$$

giving $E_2 / E_1 = 3$.

In other words, the total energy of the orbiting electron after the emission is three times as large as that before the emission.

(b) As $E_2 = \frac{-3e^2}{8\pi\varepsilon_0 r_0}$, the condition equation (i) for circular motion is no longer satisfied and the new orbit is an ellipse.
(c) Conservation of energy gives

\[ \frac{-3e^2}{8\pi\epsilon_r} = \frac{-e^2}{2\pi\epsilon_r} + \frac{m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)}{2} \]

At positions where the orbiting electron is at the distance of closest or farthest approach to the atom, we have \( \dot{r} = 0 \), for which

\[ \frac{-3e^2}{8\pi\epsilon_r} = \frac{mr^2\dot{\theta}^2}{2} - \frac{e^2}{2\pi\epsilon_r r} = \frac{J^2}{2mr^2} - \frac{e^2}{2\pi\epsilon_r r} \]

Then with

\[ J^2 = m^2\dot{\theta}^2 r_0^2 = \frac{me^2 r_0}{4\pi\epsilon_r} \]

the above becomes

\[ 3r^2 - 4r_0 r + r_0^2 = 0 \]

with solutions \( r = \frac{r_0}{3}, r = r_0 \)

Hence the distances of closest and farthest approach in the new orbit are respectively

\[ r_{\text{min}} = \frac{1}{3}, \quad r_{\text{max}} = 1 \]

**Example:** A satellite of mass \( m = 2000 \) kg is in elliptical orbit about earth. At perigee it has an altitude of 1,100 km and at apogee it has altitude 4,100 km. assume radius of the earth is \( R_e = 6,400 \) km. it is given \( GmM_e = 8 \times 10^{17} J.m \)

(a) What is major axis of the orbit?
(b) What is eccentricity of the orbit?
(c) What is angular momentum of the satellite?
(d) How much energy is needed to fix satellite in to orbit from surface of earth?
Solution: \( r_{\text{max}} = 4100 + 6400 = 10500\ km \)

\( r_{\text{min}} = 1100 + 6400 = 7500\ km \)

(a) \( r_{\text{max}} + r_{\text{min}} = 2a \Rightarrow 18000 = 2a \Rightarrow a = 9000\ km \)

\[ e = \frac{r_{\text{max}} - r_{\text{min}}}{r_{\text{max}} + r_{\text{min}}} \]

(b) \( \Rightarrow e = \frac{10500 - 7500}{10500 + 7500} = \frac{3000}{18000} = \frac{1}{6} \Rightarrow e = \frac{1}{6} \)

It is given \( k = 8 \times 10^{17}\ J \cdot m \)

(c) \( E = -\frac{k}{2a} = -\frac{8 \times 10^{17}}{18000 \times 10^3} \), \( E_f = -\frac{8 \times 10^{17}}{18} \)

\[ e = \sqrt{1 + \frac{2EJ^2}{mk^2}} \Rightarrow \left(\frac{1}{6}\right)^2 = 1 + \frac{2EJ^2}{mk^2} \]

\[ \frac{1}{36} = 1 + \frac{2EJ^2}{mk^2} \Rightarrow 140 \times 10^{26} = J^2 \Rightarrow J = \sqrt{140 \times 10^{13}} = 1.2 \times 10^{14}\ kgm / sec^2 \]

(d) When satellite is at surface of earth,

\[ R = 6400\ Km \]

\[ E_i = -\frac{GMm}{R} = -\frac{8 \times 10^{17}}{6400 \times 10^3} = -\frac{10^{17}}{800 \times 10^3} = -\frac{10^{12}}{8} = -12.5 \times 10^{10} \]

\[ E_f = -\frac{GMm}{2a} = -4.5 \times 10^{10} \ J \Rightarrow \Delta E = E_f - E_i = 8 \times 10^{10} \ J \]

**Example:** For circular and parabolic orbits in an attractive \( 1/r \) potential having the same angular momentum, show that perihelion distance of the parabola is one-half the radius of the circle.

**Solution:** For Kepler’s problem \( \frac{l}{r} = 1 + \cos \theta \), for circular orbit \( e = 0 \Rightarrow \frac{l}{r_c} = 1 \) and for parabola \( e = 1 \)

\[ \frac{l}{r} = 1 + \cos \theta \quad r_p \text{ is minimum when } \cos \theta \text{ is maximum.} \]

\[ \frac{l}{r_p} = 2 \quad \text{and} \quad \frac{l}{r_c} = 1 \Rightarrow \frac{r_p}{r_c} = \frac{1}{2} \]
Example: A planet of mass $m$ moves in the inverse square central force field of the Sun of mass $M$. If the semi-major and semi-minor axes of the orbit are $a$ and $b$, respectively, find the total energy of the planet by assuming the sun is at the center of ellipse.

Solution: Assume the Sun is at the center of elliptical orbit.

Conservation of energy:
$$\frac{1}{2}mv_1^2 - \frac{GMm}{a} = \frac{1}{2}mv_2^2 - \frac{GMm}{b}$$

Conservation of momentum:
$$L = mv_1a = mv_2b$$

$$v_2 = v_1\left(\frac{a}{b}\right)$$

$$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2 = \frac{GMm}{a} - \frac{GMm}{b} \Rightarrow \frac{1}{2}m\left(v_1^2 - v_2^2\right) = GMm\left(\frac{b-a}{ab}\right)$$

$$\frac{1}{2}mv_1^2\left(\frac{b^2-a^2}{b^2}\right) = GMm\left(\frac{b-a}{ab}\right) \Rightarrow \frac{1}{2}mv_1^2 = GMm\left(\frac{b}{a}\right)\cdot \frac{1}{b+a}$$

$$E = \frac{1}{2}mv_1^2 - \frac{GMm}{a} = GMm\frac{b}{a(b+a)} - \frac{GMm}{a}$$

$$= GMm\left(\frac{b}{(b+a)} - 1\right) = GMm\left(\frac{b-b+a}{b+a}\right) = \frac{GMm}{b+a}$$
MCQ (Multiple Choice Questions)

Q1. A planet moves round the sun. At a point \( P \), it is closest to the sun at distance \( r_1 \) and has speed \( v_1 \). At another point \( Q \), when it is farthest from the sun at a distance \( r_2 \), what is its speed?

(a) \( \frac{r_1^2 v_1}{r_2} \)  
(b) \( \frac{r_1 v_1}{r_2} \)  
(c) \( \frac{r_2^2 v_2}{r_1} \)  
(d) \( \frac{r_2 v_1}{r_1} \)

Q2. Two masses \( m \) and \( M \) are initially at rest at infinite distance apart. They approach each other due to gravitational interaction. What is their speed of approach at the instant when they are at a distance \( d \) apart?

(a) \( \left( \frac{2G(M^2 + m^2)}{d} \right)^{\frac{1}{3}} \)  
(b) \( \left( \frac{2GMm}{d(M + m)} \right)^{\frac{1}{3}} \)  
(c) \( \left( \frac{2G(M + m)}{d} \right)^{\frac{1}{2}} \)  
(d) \( \left( \frac{GMm}{d(M + m)} \right)^{\frac{1}{2}} \)

Q3. Which one of the following diagrams correctly depicts the variation of kinetic energy \( (K) \) potential energy \( (U) \) and total energy \( (E) \) of a body in circular planetary motion? \( (r \) is the radius of the circle).

(a)  
(b)  
(c)  
(d)  

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Q4. A planet of mass $m$ moves in a circular orbit of radius $r_0$ in the gravitational potential $V(r) = -\frac{k}{r}$, where $k$ is a positive constant. The orbit angular momentum of the planet is

(a) $2r_0km$  
(b) $\sqrt{2r_0km}$  
(c) $r_0km$  
(d) $\sqrt{r_0km}$

Q5. The figure given shows a satellite of mass $m$ orbiting elliptically about the centre $O$ of the Earth. The mass of the Earth is $M$. The work done on the satellite as it moves from $A$ to $B$ in its orbit is

(a) $\frac{GMr_B}{r_A} \left(1 - \frac{r_B}{r_A} \right)$  
(b) $\frac{GMr_A}{r_B} \left(1 - \frac{r_A}{r_B} \right)$  
(c) $GMM \left(1 - \frac{r_B}{r_A} \right)$  
(d) $\frac{GMr_B}{r_A^2 - r_B^2}$

Q6. If the gravitational force is assumed to vary inversely as the $n$th power of distance $r$, then how does the time period of a planet the sun depend upon $r$?

(a) $r^n$  
(b) $r^{-n}$  
(c) $r^{\left(n+1\right)/2}$  
(d) $r^{\left(n-1\right)/2}$

Q7. The figure below shows the motion of a planet around the sun in an elliptical orbit with sun as the focus. Areas $SOA$ and $SAB$ shown in the figure can be assumed to be equal. If $t_1$ and $t_2$ represent the times for the planet to move from $O$ to $A$ and $A$ to $B$ respectively; then

(a) $t_1 < t_2$  
(b) $t_1 > t_2$  
(c) $t_1 = t_2$  
(d) None of these

Q8. If $R$ is the radius of the earth $\rho$ is mean density and $G$ the gravitational constant, then the earth’s surface potential will be nearly equal to

(a) $\frac{\pi \rho G}{R^2}$  
(b) $\frac{4}{3} \pi R^3 \rho G$  
(c) $\frac{4}{3} \pi \rho G$  
(d) $\frac{4}{3} \pi \rho G$

Q9. Three objects $S_1, S_2$ and $S_3$ having same mass $m$ are in different elliptic orbits of same semi-major axis, about an object of mass $M$. If the eccentricities of the orbits are $e_1, e_2$ and $e_3$ respectively such that $e_1 < e_2 < e_3$ and if $E_1, E_2$ and $E_3$ are their respective mechanical energies, then which one of the following is correct?

(a) $E_1 < E_2 < E_3$  
(b) $E_1 > E_2 > E_3$  
(c) $E_1 = E_2 = E_3$  
(d) $E_1, E_2$ and $E_3$ cannot be compared
Q10. A particle is moving in an inverse square field. If the total energy of the particle is positive, then trajectory of particle is
(a) circular  (b) elliptical  (c) parabolic  (d) hyperbolic

Q11. In a central force field, the trajectory of a particle of mass \( m \) and angular momentum \( J \) in plane polar coordinates is given by,

\[
\frac{1}{r} = \frac{m}{J^2} (1 + \varepsilon \cos \theta)
\]

where, \( \varepsilon \) is the eccentricity of the particle’s motion. Which one of the following choice for \( \varepsilon \) gives rise to a parabolic trajectory?
(a) \( \varepsilon = 0 \)  (b) \( \varepsilon = 1 \)  (c) \( 0 < \varepsilon < 1 \)  (d) \( \varepsilon > 1 \)

Q12. Match the following List I with List II.

<table>
<thead>
<tr>
<th>List I</th>
<th>List II</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Eccentricity ( e &gt; 0 ), energy ( E &gt; 0 )</td>
<td>1. Circular orbit</td>
</tr>
<tr>
<td>B. Eccentricity ( e = 1 ) ( E = 0 )</td>
<td>2. Elliptical orbit</td>
</tr>
<tr>
<td>C. ( e &lt; 1 ), ( E &lt; 0 )</td>
<td>3. Parabolic orbit</td>
</tr>
<tr>
<td>D. ( e = 0 ), ( E = -\frac{mk^2}{2J^2} )</td>
<td>4. Hyperbolic orbit</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>(b)</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>(c)</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>(d)</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Q13. A satellite in a circular orbit about the earth has a kinetic energy \( E_K \). What is the minimum amount of energy to be added, so that it escape from the earth?

(a) \( \frac{E_K}{4} \)  (b) \( \frac{E_K}{2} \)  (c) \( E_K \)  (d) \( 2E_K \)
Q14. Two particles of identical mass move in circular orbits under a central potential \( V(r) = \frac{1}{2} kr^2 \). Let \( l_1 \) and \( l_2 \) be the angular momenta and \( r_1, r_2 \) be the radii of the orbits respectively. If \( \frac{J_1}{J_2} \), the value of \( \frac{r_1}{r_2} \) is:

(a) \( \sqrt{2} \)  
(b) \( \frac{1}{\sqrt{2}} \)  
(c) 2  
(d) \( \frac{1}{2} \)

Q15. A planet of mass \( m \) moves in the gravitational field of the Sun (mass \( M \)). If the semi-major and semi-minor axes of the orbit are \( a \) and \( b \) respectively, the angular momentum of the planet is

(a) \( \sqrt{2Gm^2(a+b)} \)  
(b) \( \sqrt{2Gm^2(a-b)} \)  
(c) \( \frac{2Gm^2ab}{a-b} \)  
(d) \( \frac{2Gm^2ab}{a+b} \)

NAT (Numerical Answer Type)

Q16. If a planet revolves round the sun in a circular orbit of radius \( a \) with a period of revolution \( T \), then \( k \) being a positive constant if \( T \propto a^\alpha \) then value of \( \alpha \) is given by ……….. 

Q17. Consider a satellite going round the earth in a circular orbit at a height of \( 2R \) from the surface of the earth, where \( R \) is the radius of the earth. The speed of the satellite is \( \left( \frac{gR}{\beta} \right)^\alpha \) then value of \( \alpha \)……….. and value of \( \beta \) is ………….. 

Q18. A body of mass \( m \) moves under the action of a central force with potential \( V(r) = Ar^3 \) \( (A > 0) \) where \( r \) about the origin. Then kinetic energy will the orbit be a circle of radius \( R \) is \( \alpha AR^3 \) what is value of \( \alpha \)……….. ?

Q19. A comet moves in an elliptical orbit with an eccentricity of \( e = 0.20 \) around a star. The distance between the perihelion and the aphelion is \( 1.0 \times 10^6 \) km. If the speed of the comet at perihelion \( 60 \) km/s then the speed of the comet at the aphelion is……….. km/s

Q20. The acceleration due to gravity \( (g) \) on the surface of Earth is approximately 2.6 times that on the surface of Mars. Given that the radius of Mars is about one half the radius of Earth, the ratio of the escape velocity on Earth to that on Mars is approximately …………..
MSQ (Multiple Select Questions)

Q21. A satellite is in elliptical orbit the earth (radius = 6400 km). At perigee it has an altitude of 1100 km and the apogee its apogee its altitude is 4100 km.
(a) Then The major axis of the orbit is 18,000 km
(b) Then The major axis of the orbit is 9,000 km
(c) The eccentricity is 0.16
(d) Eccentricity is 0.33

Q22. A planet (mass $M_p$) is moving around the sun (mass $M_s$) in an elliptical orbit of semimajor axis $a$. Total energy of the planet is
(a) $\frac{GM_p M_s}{4a}$
(b) $\frac{2GM_p M_s}{a}$
(c) $\frac{GM_p M_s}{2a}$
(d) $\frac{GM_p M_s}{2a}$

Q23. Consider the following statements for a particle moving in an elliptic orbit under the influence of a central force.
(a) The radius vector covers equal area in equal time.
(b) The motion takes place in a plane.
(c) The angular momentum is a constant of motion.
(d) The energy remains constant and negative

Q24. Consider the following expressions in respect of earth’s motion around the sun in a circular orbit of radius $r$ with a linear speed $v$ and angular speed $\omega$. Then which of the following expressions are correct?
(a) $v$ is proportional to $r^{-\frac{1}{2}}$
(b) $v$ is proportional to $r^{-\frac{1}{2}}$
(c) $\omega v$ is proportional to $r^{-\frac{5}{2}}$
(d) $\omega$ is proportional to $r^{-\frac{3}{2}}$
Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (b)
Solution: closest from the sun = \( r_1 \), speed = \( v_1 \)

Farthest from the sun = \( r_2 \)

According to conservation of momentum \( mr_1v_1 = mr_2v_2 \)

\[ v_2 = \frac{r_1v_1}{r_2} \]

Ans. 2: (c)
Solution: if reduce mass \( \mu = \frac{mM}{m+M} \) then \[ \frac{1}{2} \mu v^2 = 0 - \left( -\frac{GMm}{d} \right) \] \[ \Rightarrow v^2 = \frac{2G(M+m)}{d} \] (acceleration due to gravity upward from the earth)

\[ v = \sqrt{\frac{2G(M+m)}{d}} \]

Ans. 3: (a)
Solution: For circular motion

\[ \frac{mv^2}{r} = \frac{k}{r^2} \Rightarrow K = \frac{mv^2}{2} = \frac{k}{r^2} \]

\[ E = K + U \]

\[ U = -\frac{k}{r} \]

\[ E = -\frac{k}{2r} \]

Ans. 4: (d)
Solution: \( V_{\text{effective}} = \frac{J^2}{2mr^2} - \frac{k}{r} \)

\[ \Rightarrow \frac{dV_{\text{effective}}}{dr} = -\frac{J^2}{mr^3} + \frac{k}{r^2} = 0 \text{ at } r = r_0 \]

so \[ J = \sqrt{k rm} \]

Ans. 5: (a)
Solution: By conservation law of energy

\[ \Rightarrow dW = \left( PE \right)_A - \left( PE \right)_B \quad \text{[ By equation (i)]} \]

\[ = \left( -\frac{GMm}{r_A} \right) - \left( -\frac{GMm}{r_B} \right) = \frac{GMm}{T_B} - \frac{GMm}{T_A} = \frac{GMm}{r_B} \left[ 1 - \frac{r_B}{r_A} \right] \]
Ans. 6: (c)

Solution: If a planet revolves with velocity $v$ radius $r$ then the centripetal force it gives as

$$F = \frac{mv^2}{r} \quad (i)$$

The attractive $M$ between sun and planet is given as

$$F = \frac{k}{r^n} \quad (ii)$$

$k$ being a constant

By equations (i) and (ii) $\frac{k}{r^n} = \frac{mv^2}{r} \Rightarrow v^2 = \frac{k}{mr^{n-1}} \Rightarrow v = \left(\frac{k}{mr^{n-1}}\right)^{1/2}$

Hence, time period

$$\frac{2\pi r}{v} = 2\pi r \left(\frac{k}{mr^{n-1}}\right)^{1/2} = \frac{2\pi r r^{n-1/2}}{\sqrt{k}} = 2\pi \sqrt{\frac{m}{kr^{n+1}}}$$

Hence, time period is proportional to $r^{-\frac{n+1}{2}}$

Ans. 7: (c)

Solution: Kepler's second law states that the areal velocity swept out by a radius drawn from sun to a planet is constant, i.e.,

$$\frac{dA}{dt} = \text{constant}$$

Ans. 8: (c)

Solution: Potential at earth surface $V = -\frac{GM_e}{R_e}$

Mass of the earth $M_e = \text{volume} \times \text{density} = \frac{4}{3}\pi R_e^3 \rho$ \hspace{1cm} (ii)

By equation (i) and (ii) becomes

$$V = -\frac{G}{R_e} \left(\frac{4}{3}\pi R_e^3 \rho\right) = -\frac{4}{3}\pi R_e^2 \rho G$$
Ans. 9: (b)
Solution: The eccentricity of the ellipse is given as
\[ e = \sqrt{1 + \frac{2EJ^2}{Mk^2}} \]
where \( k \) = constant
\( M \) = mass of the object
\( E \) = mechanical energy of the object eccentricity which is negative
But \( e_1 < e_2 < e_3 \) given in question
Then by equation (iii), \( E_1 > E_2 > E_3 \)

Ans. 10: (c)
Solution: \( e = \left(1 + \frac{2EJ^2}{mk^2}\right)^{\frac{1}{2}} \Rightarrow e > 1 \quad \therefore \left(\frac{J}{k}\right)^2 \) is positive

Ans. 11: (b)
Solution: \( \frac{J}{r} = \frac{m}{J^2} \left(1 + e \cos \theta\right) \) for parabolic trajectory \( e = 1 \).

Ans. 12: (d)

Ans. 13: (c)
Solution: The potential energy of the satellite at a height \( h \), i.e., at distance \((R_e + h)\) from the centre of the earth is given as
\[ PE = -\frac{GM_em}{r} \Rightarrow PE = -\frac{GM_em}{R_e + h} \]
where \( m \) is mass of satellite and \( M_e \) is mass of earth.

\[ \frac{mv^2}{r} = \frac{GM_em}{r^2} \Rightarrow mv^2 = \frac{GM_em}{r} \Rightarrow KE = \frac{1}{2}mv^2 = \frac{GM_em}{2r} \Rightarrow KE = \frac{GM_em}{2r} \]
But equations, (i) and (ii), we get
\[ KE = -\frac{1}{2}PE \Rightarrow PE = -2(KE) \Rightarrow PE = -2E_K \quad \text{(given } KE = E_K \text{)} \]
The total energy of satellite = \( PE + KE = -2E_K + E_K = -E_K \)
The total energy \((KE + PE)\) of satellite is negative and is called the binding energy of the satellite.

Satellite becomes free, if its total energy is non negative

Hence, in order to escape from the earth, the minimum amount of energy to be supplied is \(E_K\)

Ans. 14: (a)

Solution: 
\[
V_{eff} = \frac{J^2}{2mr^2} + \frac{1}{2} kr^2
\]
where \(J\) is angular momentum.

Condition for circular orbit 
\[
\frac{\partial V_{eff}}{\partial r} = 0 \Rightarrow -\frac{J^2}{mr^4} + kr = 0 \Rightarrow J^2 \propto r^4 \Rightarrow J \propto r^2.
\]

Thus \(J_2 = \left(\frac{r_1}{r_2}\right)^2 \Rightarrow \frac{J_1}{J_2} = \frac{r_1}{r_2} = \frac{\sqrt{J_1}}{\sqrt{J_2}} \Rightarrow \frac{J_1}{J_2} = 2\) since \(\frac{J_1}{J_2} = 2\).

Ans. 15:

Solution: Assume Sun is at the centre of elliptical orbit.

Conservation of energy
\[
\frac{1}{2} mv_1^2 - \frac{GMm}{a} = \frac{1}{2} mv_2^2 - \frac{GMm}{b}
\]

Conservation of momentum
\[
L = mv_1a = mv_2b
\]

\[
v_2 = v_1 \left(\frac{a}{b}\right)
\]

\[
\frac{1}{2} mv_1^2 - \frac{1}{2} mv_2^2 = \frac{GMm}{a} - \frac{GMm}{b} \Rightarrow \frac{1}{2} m \left(v_1^2 - v_2^2 \frac{a^2}{b^2}\right) = GMm \left(\frac{b-a}{ab}\right)
\]

\[
\frac{1}{2} \frac{mv_1^2 \left(b^2-a^2\right)}{b^2} = GMm \left(\frac{b-a}{ab}\right) \Rightarrow \frac{1}{2} mv_1^2 = GMm \left(\frac{b}{a}\right) \cdot \frac{1}{(b+a)}
\]

\[
v_1 = \sqrt{2GM \left(\frac{b}{a}\right) \cdot \frac{1}{(b+a)}}
\]

\[
L = mv_1a = m \sqrt{2GM \left(\frac{b}{a}\right) \cdot \left(\frac{1}{b+a}\right)} \cdot a = m \sqrt{2GMab} \Rightarrow L = \sqrt{\frac{2GMm^2ab}{a+b}}
\]
NAT (Numerical Answer Type)

Ans. 16: 1.5

Solution: The Kepler’s third law states that square of time perk planet is directly proportional to the cube of the set axis $a$

i.e., $T^2 \propto a^3$

$T \propto a^\frac{3}{2}$

Ans. 17: $\alpha = 0.5, \beta = 3$

Solution: Centripetal force $= \frac{mv^2}{r}$

Newton’s law of gravitation $= \frac{GMm}{r^2}$

From equation (i) and (ii), in equilibrium

$\frac{mv^2}{r} = \frac{GMm}{r^2}$

$v = \sqrt{\frac{GM}{r}}$

$g = \frac{GM}{R^2}$ (acceleration due to gravity)

$GM = gR^2$

$v = \sqrt{\frac{gR^2}{r}} = \sqrt{\frac{gR^2}{R+2R}} = \sqrt{\frac{gR^2}{3R}} = \left(\frac{gR}{3}\right)^{1/2}$

Ans. 18: 1.5

Solution: If $U$ is potential energy then applied force $F$ is given by

$F = -\frac{dU}{dr} = -\frac{d}{dr}(Ar^3)$ at $r = R$

$\Rightarrow F = -3AR^2$ [Since]

This force is equal to centripetal force $= \frac{mv^2}{R}$
i.e. \[ \frac{mv^2}{R} = F \Rightarrow \frac{mv^2}{R} = 3MAR^2 \Rightarrow mv^2 = 3AR^3 \]
\[ \frac{1}{2}mv^2 = \frac{3}{2}AR^3 \Rightarrow KE = \frac{3}{2}AR^3 \]

Ans. 19: 40

Solution: According to conservation of momentum is \( mv_p r_p = mv_a r_a \)

\[ 2a = 10km \quad a = 5km \quad r_p = a(1-e) = 4km \quad r_a = a(1+e) = 6 \]
\[ v_a = \frac{r_p v_p}{r_a} = \frac{4 \times 60}{6} = 40 km/s \]

Ans. 20: 2.3

Solution: Escape velocity = \( \sqrt{2gR} \)

\[ \frac{\text{Escape velocity of Earth}}{\text{Escape velocity of Mass}} = \sqrt{\frac{g_R}{g_m}R_e} = 2.3 \quad \text{where} \quad \frac{R_e}{R_m} = 2 \quad \text{and} \quad \frac{g_e}{g_m} = 2.6. \]

**MSQ (Multiple Select Questions)**

Ans. 21: (a) and (c)

Solution: Major axis \( 2a = \) maximum distance + minimum distance

Maximum distance = maximum altitude + radius of earth = \((4100 + 6400)\)

Minimum distance from centre of earth = 6400 + 1100

So, \( 2a = (4100 + 6400) + (6400 + 1100) = 18000 \) km

\[ e = \frac{r_{max} - r_{min}}{r_{max} + r_{min}} = \frac{1}{6} \]

Ans. 22: (d)

Solution: \( E = -\frac{k}{2a} = -\frac{GM_p M_s}{2a} \)

Ans. 23: (a), (b), (c) and (d)

Solution: Angular momentum \( J = r \times mv \) implies that angular momentum is perpendicular to \( r \) and \( v \).

Hence, the path of particle under the influence of a central force must lie in a plane.
Torque \( \tau = \frac{dJ}{dt} = r \times \vec{F} = r \times rf(r) = 0 \)

Thus, angular momentum \( J = \text{constant} \)

Areal velocity \( \frac{1}{2} \dot{r} \times \vec{v} = \frac{1}{2} \times \frac{J}{m} = \frac{J}{2m} = \text{constant} \quad (J = mv) \)

The radius vector of orbit sweeps out equal area in equal interval of time.

Ans. 24: (a) and (d)

Solution: It is supposed that earth revolves around the sun in orbit of radius \( r \) with linear velocity \( v \)

Centripetal force \( = \frac{mv^2}{r} \) \( \quad (i) \)

Where \( m = \text{mass of the earth.} \)

The attractive force towards sun is given as

\[ F = \frac{GM_1m}{r^2} \quad (ii) \]

\( M_1 = \text{mass of the sun} \)

by equation (i) and (ii)

\[ \frac{GM_1m}{r^2} = \frac{mv^2}{r} \]

\[ \Rightarrow v^2 = \frac{GM_1}{r} \]

\[ \Rightarrow v = \sqrt{\frac{GM_1}{r}} \quad (iii) \]

Since \( G = \text{gravitational constant hence,} \quad \Rightarrow v \propto r^{-\frac{1}{2}} \quad (iv) \)

First statement is true.

If \( \omega \) is angular velocity, then

\[ v = \omega r \Rightarrow \omega = \frac{v}{r} \Rightarrow \omega = \left( \frac{\sqrt{GM_1}}{r} \right) \frac{1}{r} \Rightarrow \omega \propto r^{-\frac{3}{2}} \quad (v) \]

Hence equation (i) \( v \propto r^{-\frac{1}{2}} \) \( \quad \) (ii) \( \omega \propto r^{-\frac{3}{2}} \)
4. Moment of Inertia Rotational Dynamics

4.1 Rigid Body Dynamics

A rigid body is defined as system of particles in which the distance between any two particles remains fixed throughout the motion. Degree of freedom of rigid body.

To define rigid body there must be minimum 3 non-collinear point.

Let \( p_1(x_1, y_1, z_1), \ p_2(x_2, y_2, z_2), \ p_3(x_3, y_3, z_3) \) is there three point.

So equation of constrained is

\[
\begin{align*}
 r_{12} &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = c_1 \\
 r_{23} &= (x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 = c_2 \\
 r_{13} &= (x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2 = c_3 \\
 \text{Dof} &= 3N - k \\
 3 \times 3 - 3 &= 6 \quad \text{So there is six degree of freedom for rigid body}
\end{align*}
\]

4.1.1 Angular Momentum and Inertia Tensor.

Let us consider the motion of a rigid body rotating about a fixed point \( o \) in the body as shown in figure.

At any instant of time, the body will rotating with velocity \( \omega \) about the instantaneous through \( o \).

A particle \( P \) of the body, having the position vector \( r_1 \) with respect to \( o \) has instantaneous velocity \( v_i \) relative to \( o \), given by

\[
\begin{align*}
 \vec{v}_i &= \dot{\vec{r}}_i \\
 \dot{\vec{r}} &= x\dot{i} + y\dot{j} + z\dot{k}
\end{align*}
\]

Angular momentum of Point \( P \) about the point \( o \) is given by

\[
\begin{align*}
 \vec{J}_p &= \vec{r}_i \times m_i \vec{v}_i \\
 J_p &= \vec{r}_i \times m_i (\dot{\omega} \times \vec{r}_i)
\end{align*}
\]
\[ J = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_i = \sum_{i=1}^{N} m_i \mathbf{r}_i \times (\mathbf{\omega} \times \mathbf{r}_i) \]

\[ J = \sum_{i} m_i \left[ r_i^2 \mathbf{\omega} - (r_\mathbf{\omega} r_i) \right] \]

\[
\begin{pmatrix}
J_x \\
J_y \\
J_z
\end{pmatrix} =
\begin{pmatrix}
I_{xx} & I_{xy} & I_{xz} \\
I_{yx} & I_{yy} & I_{yz} \\
I_{zx} & I_{zy} & I_{zz}
\end{pmatrix}
\begin{pmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{pmatrix}
\]

\[ I_\alpha = \sum_{\beta} I_{\alpha\beta} \omega_\beta \quad \text{\(I_{\alpha\beta}\) is inertia tensor} \]

where \( I_{xx} = \sum_{i} m_i (y_i^2 + z_i^2) \), \( I_{yy} = \sum_{i} m_i (x_i^2 + z_i^2) \), \( I_{zz} = \sum_{i} m_i (x_i^2 + y_i^2) \) known as moment of inertia and \( I_{xy} = I_{yx} = -\sum_{i} m_i x_i y_i \), \( I_{xz} = I_{zx} = -\sum_{i} m_i x_i z_i \), \( I_{yz} = I_{zy} = -\sum_{i} m_i y_i z_i \) is known as product of inertia.

\[ I_{\alpha\beta} = I_{\beta\alpha} = \sum_{i=1}^{N} m_i \left[ \delta_{\alpha\beta} r_i^2 - x_\alpha x_\beta \right] \]

where \( \alpha = 1, 2, 3 \), \( \beta = 1, 2, 3 \) and one can denote \( x, y, z \) by \( x_1, x_2, x_3 \) respectively.

For continuous system \( I_{\alpha\beta} \) is reduce to

\[ \int dm \left[ \delta_{\alpha\beta} x_i^2 - x_\alpha x_\beta \right] \] where \( dm \) is elemental mass.
4.1.2 Principal Moment of Inertia:

If one can diagonalized Inertia tensor into diagonal matrix then the diagonal element is known as principal moments of inertia

and \( x, y, z \) component of angular momentum reduce to

\[
J_x = I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z \\
J_y = I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z \\
J_z = I_{zx} \omega_x + I_{zy} \omega_y + I_{zz} \omega_z
\]

\[
\begin{pmatrix}
I_{xx} - I & I_{xy} & I_{xz} \\
I_{yx} & I_{yy} - I & I_{yz} \\
I_{zx} & I_{zy} & I_{zz} - I
\end{pmatrix}
\begin{pmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{pmatrix} = 0
\]

characteristic equation \( I = I_1, I_2, I_3 \) are three principal moment of inertia.

Rotational kinetic energy of a rigid body

\[
\vec{\bar{v}}_i = \vec{\omega} \times \vec{r}_i
\]

**Kinetic energy of particle** \( m_i \) (which is explained previous topic).

\[
T_i = \frac{1}{2} m_i \vec{v}_i^2 = \frac{1}{2} m_i \vec{\bar{v}}_i \cdot \vec{\bar{v}}_i
\]

Kinetic energy for entire body

\[
T = \sum_i \frac{1}{2} m_i \vec{v}_i^2 = \sum_i \frac{1}{2} m_i \vec{\bar{v}}_i \cdot \vec{\bar{v}}_i = \frac{1}{2} \sum_i \vec{\omega} \cdot (\vec{r}_i \times m \vec{\bar{v}}_i) = \frac{1}{2} \vec{\omega} \cdot \sum_i (\vec{r}_i \times m \vec{\bar{v}}_i)
\]

\[
J = \sum_i (\vec{r}_i \times m \vec{\bar{v}}_i) \text{ so } T = \frac{1}{2} \vec{\omega} \cdot \vec{J}
\]

Kinetic energy in a co-ordinate system of principle axis is given by

\[
T = \frac{1}{2} I_x \omega_x^2 + \frac{1}{2} I_y \omega_y^2 + \frac{1}{2} I_z \omega_z^2
\]
Example: Consider a cube of volume $a^3$ and mass $M$ which is situated such the origin $o$ is at one of the corner consider cube has uniform density $\rho$.

(a) Find the moment of inertia tensor.

(b) Find the principal moment of inertia tensor.

Solution: (a) $I_{xx} = \int \int \int \rho \left( y^2 + z^2 \right) dV = \int_0^a \int_0^a \int_0^a \rho \left( y^2 + z^2 \right) dx dy dz = \frac{2\rho a^5}{3}$

$\Rightarrow I_{xx} = \frac{2Ma^5}{a^3} = \frac{2}{3}Ma^2$ where $\rho = \frac{M}{a^3}$

Similarly,

$I_{yy} = \frac{2}{3}Ma^2$ and $I_{zz} = \frac{2}{3}Ma^2$

$I_{xy} = I_{yx} = -\frac{Ma^2}{4}$ and $I_{xz} = I_{zx} = -\frac{Ma^2}{4}$

$I_{yz} = I_{zy} = -\frac{Ma^2}{4}$

Hence, $I = \frac{Ma^2}{12} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$.

(b) Principle M.I. is given by characteristic equates

$\begin{pmatrix} 8Ma^2 - I & -3Ma^2 & 3Ma^2 \\ -3Ma^2 & 8Ma^2 - I & 3Ma^2 \\ 3Ma^2 & 3Ma^2 & 8Ma^2 - I \end{pmatrix} = 0$

$\Rightarrow I_1 = I_2 = \frac{11}{12}Ma^2$ and $I_3 = \frac{1}{6}Ma^2$
4.1.3 Theorem of Parallel Axes:

Suppose we have to obtain the moment of inertia of a body about a given line AB. Let C be the centre of mass of the body and CZ be the line parallel to AB through C. Let \( I \) and \( I_c \) be the moment of inertia of the body about AB and CZ respectively. Then

\[
I = I_c + md^2
\]

where \( d \) is the perpendicular distance between lines \( AB \) and \( CZ \) and \( m \) is mass of the body.

4.1.4 Theorem of Perpendicular Axes:

If \( I_x, I_y \) and \( I_z \) is principle moment of inertia about \( x, y, z \) axis respectively, then

\[
I_x + I_y = I_z
\]

4.2 Particular Cases of Moment of Inertia

Let us now see how to obtain expressions for moments of inertia in some typical and important cases, with regular geometrical shapes.

4.2.1. Moment of Inertia of a Uniform Rod

(i) about an axis through its centre and perpendicular to its length.

Let \( AB \) be a thin uniform rod, of length \( l \) and mass \( M \), free to rotate about an axis \( YOY' \) passing through its centre \( O \) and perpendicular to its length, as shown in figure. Since the rod is uniform, its mass per unit length is clearly \( M/l \).

Considering a small element of the rod, of length \( dx \) at a distance \( x \) from the axis through \( O \), we have mass of the element \( = (M/l) \cdot dx \) and therefore, its moment of inertia about the axis \( (YOY') \) through \( O = (M/l) \cdot dx \cdot x^2 \).
The moment of inertia \( I \) of the whole rod about the axis \( YOY' \) is thus given by the integral of the above expression between the limits \( x = -l/2 \) and \( x = +l/2 \) or by twice its integral between the limits \( x = 0 \) and \( x = l/2 \), i.e.

\[
I = 2 \int_{-l/2}^{l/2} \frac{M}{l} x^2 \, dx = \frac{2M}{l} \int_{0}^{l/2} x^2 \, dx = \frac{2M}{l} \left[ \frac{x^3}{3} \right]_{0}^{l/2} = \frac{2M}{l} \cdot \frac{l^3}{24} = \frac{Ml^2}{12}
\]

(ii) About an axis through one end of the rod and perpendicular to its length.

Proceeding as in case (i) above, we obtain the moment of inertia of the rod about the axis, now passing through one end \( A \) of the rod, by integrating the expression for the moment of inertia of the element \( dx \) of the rod, between the limits \( x = 0 \) at \( A \) and \( x = l \) at \( B \), i.e.,

\[
I = \int_{0}^{l} \frac{M}{l} x^2 \, dx = \frac{2M}{l} \cdot \frac{l^3}{24} = \frac{Ml^2}{12}
\]

Alternatively, we could obtain the same result by an application of the principle of parallel axes, according to which

M.I. of the rod about the axis \( YAY' \) = its M.I. about a parallel axis through \( O \) + (mass of the rod \( \times \) square of the distance between the two axes).

So that, \( I = \frac{Ml^2}{12} + M \left( \frac{l}{2} \right)^2 = \frac{Ml^2}{12} + \frac{Ml^2}{4} = \frac{Ml^2}{3} \)

4.2.2 Moment of Inertia of a Rectangular Lamina (or bar)

(i) About an axis through its centre and parallel to one side. Let \( ABCD \) be a rectangular lamina, of length \( l \), breadth \( b \) and mass \( M \) and let \( YOY' \) be the axis through its centre \( O \) and parallel to the side \( AD \) or \( BC \) about which its moment of inertia is to be determined.
Consider an element, or a small rectangular strip of the lamina, parallel to, and at a distance \( x \) from the axis. The area of this strip or element is \( dx \times b \). And, since the mass per unit area of the lamina is \( \frac{M}{l \times b} \), we have

mass of the strip or element = \( \frac{M}{l \times b} \times dx \times b = \frac{M}{l} \times dx \).

And therefore, M.I. of the element about the axis \( YOY' \)

\[ I = \frac{M}{l} \times dx \cdot x^2 \]

The moment of inertia \( I \) of the whole rectangular lamina is then given by twice the integral of the above expression between the limits \( x = 0 \) and \( x = \frac{l}{2} \).

i.e.,

\[ I = 2 \int_{0}^{\frac{l}{2}} \frac{M}{l} x^2 dx = \frac{2M}{l} \int_{0}^{\frac{l}{2}} x^2 dx = \frac{2M}{l} \left[ \frac{x^3}{3} \right]_{0}^{\frac{l}{2}} \]

\[ = \frac{2M}{l} \cdot \frac{l^3}{24} = \frac{Ml^2}{12} \]

As will be readily seen, if \( b \) be small, the rectangular lamina becomes a rod of length \( l \) whose M.I. about the axis \( YOY' \) passing through its centre and perpendicular to its length would be \( \frac{Ml^2}{12} \), as obtained under case I, (i) above.

(ii) About one side. in this case, since the axis coincides with \( AD \) or \( BC \), we integrate the expression for the M. I. of the element of length \( dx \) and distance \( x \) from the axis, i.e.

\( \left( \frac{M}{l} \right) dx \cdot x^2 \) between the limits \( x = 0 \) at \( AD \) and \( x = l \) at \( BC \). So that M. I. of the lamina about side \( AD \) or \( BC \) is given by

\[ I = \frac{1}{3} \int_{0}^{l} \frac{M}{l} x^2 dx = \frac{M}{l} \left[ \frac{x^3}{3} \right]_{0}^{l} = \frac{Ml^2}{3} \]
Alternatively, by the principle of parallel axes, \( I = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{3} \).

This is again the same case as that of the M.I. of a rod about an axis passing through one of its ends and perpendicular to its length [case I (ii)], for, as pointed out above, if \( b \) be small, the rectangular lamina too reduces to a thin rod of length \( l \).

(iii) About an axis passing through its centre and perpendicular to its plane. This may be easily obtained by an application of the principle of perpendicular axes to case (i) above, according to which, M. I. of the lamina about an axis through \( O \) and perpendicular to its plane = M.I. of the lamina about an axis through \( O \) parallel to \( b \) + M.I. of the lamina about an axis through \( O \) parallel to \( l \) i.e.,

\[
I = \frac{Ml^2}{12} + \frac{Mb^2}{12} = \frac{M\left(l^2 + b^2\right)}{12}
\]

This relation is true for both thin (i.e., laminar) as well as thick plates or bars, because no stipulation has been made in deducing it as to the thickness of the lamina. And, in fact, a thick rectangular plate or bar may be regarded as a combination of thin or laminar plates or bars, piled up one over the other.

(iv) About an axis passing through the mid-point of one side and perpendicular to its plane. In this case the axis passes through the mid-point of side \( AB \) or \( BC \), say, and perpendicular to the plane of the lamina, so that it is parallel to the axis through \( O \) (the c.m. of the lamina) in case (iii).

In accordance with the principle of parallel axes, therefore, the M. I. of the lamina about this axis is given by

\[
I = M\left(\frac{l^2 + b^2}{12}\right) + M\left(\frac{l}{2}\right)^2 = M\left(\frac{l^2 + b^2}{12} + \frac{l^2}{4}\right) = M\left(\frac{l^2 + b^2}{3}\right)
\]

And, if the axis passes through the mid-point of \( AB \) or \( DC \), we, similarly have

\[
I = M\left(\frac{l^2 + b^2}{12}\right) + M\left(\frac{b}{2}\right)^2 = M\left(\frac{l^2 + b^2}{12} + \frac{b^2}{4}\right) = M\left(\frac{l^2 + b^2}{3}\right)
\]
(v) About an axis passing through one of its corners and perpendicular to its plane. Let the axis pass through the corner \( D \) of the lamina. Since it is perpendicular to the plane of the lamina, it is parallel to the axis through its centre of mass \( O \) in case (iii). Again, therefore, by the principle of parallel axes, we have moment of inertia of the rectangular lamina about this axis through \( D \) given by

\[
I = M \left( \frac{l^2 + b^2}{12} \right) + Mr^2,
\]

where \( r \) is the distance between the two axes. Clearly, \( r^2 = (l/2)^2 + (b/2)^2 = (l^2 + b^2)/4 \).

So that, \( I = M \left( \frac{l^2 + b^2}{12} \right) + M \left( \frac{l^2 + b^2}{4} \right) = M \left( \frac{l^2 + b^2 + 3l^2 + 3b^2}{12} \right) = M \left( \frac{l^2 + b^2}{3} \right) \).

4.2.3 Moment of inertia of a hoop or a thin circular ring.

(i) About an axis through its centre and perpendicular to its plane. Let the radius of the hoop or the thin circular ring be \( R \) and its mass \( M \).

Consider a particle of mass \( m \) of the hoop or the ring. Clearly, its M.I. about an axis through the centre \( O \) of the hoop or the ring and perpendicular to its plane = \( mR^2 \).

\[ \therefore \text{M.I. of the entire hoop or ring about this axis passing through its centre and perpendicular to a plane, } I = \Sigma mR^2 = MR^2 \text{ [ } \Sigma m = M \text{, the mass of the hoop or ring} \]  

(ii) About its diameter. Obviously, due to symmetry, the M.I. of the hoop or the ring will be the same about one diameter as about another. Thus, if \( l \) be its M.I. about the diameter \( XOX' \) (Figure), it will also be \( I \) about the diameter \( YOY' \) perpendicular to \( XOX' \).

By the principle of perpendicular axes, therefore, the M.I. of the hoop or the ring about the axis through its centre \( O \) and perpendicular to its plane is equal to the sum of its moments of inertia also the perpendicular axes \( XOX' \) and \( YOY' \) in its own plane and intersecting at \( O \), i.e.,

\[ I + I = MR^2 \text{ or } 2I = MR^2 \text{ whence, } I = \frac{MR^2}{2}. \]
4.2.4 Moment of inertia of a circular lamina or disc

(i) About an axis through its centre and perpendicular to its plane:

Sine as we know, \( I = MK^2 = MR^2 \), we have here \( MK^2 = MR^2 \) whence, \( K = R \), i.e., the radius of the hoop or the ring is equal to its radius of gyration about the axis through its centre and perpendicular to its plane. This gives us another definition of the radius of gyration of a body about a given axis, viz., that it is numerically equal to the radius of a hoop or a circular ring, infinitely thin, of the same mass as the body and having the same moment of inertia about the axis passing through its centre and perpendicular to its plane as the body has about the given axis.

Let \( M \) be the radius of the disc and \( R \), its radius, so that its mass per unit area is equal to \( \frac{M}{\pi R^2} \).

Considering a ring of the disc, of width \( dx \), and distance \( x \) from the axis passing through \( O \) and perpendicular to the plane of the disc, we have

\[
\text{Area of the ring} = \text{circumference} \times \text{width} = 2\pi x dx,
\]

and hence its mass

\[
\frac{M}{\pi R^2} \times 2\pi x dx = \frac{2Mx dx}{R^2}.
\]

Since the whole disc may be supposed to be made up of such like concentric rings of radius ranging from 0 to \( R \), the M.I. of the whole disc about the axis through \( O \) and perpendicular to its plane, i.e. \( I \), obtained by integrating the above expression for the M.I. of the ring, between the limits \( x = 0 \) and \( x = R \). Thus,

\[
I = \int_0^R \frac{2Mx}{R^2} x^2 dx = \frac{2M}{R^2} \left[ \frac{x^4}{4} \right]_0^R = \frac{2M}{R^2} \cdot \frac{R^4}{4} = \frac{MR^2}{2}.
\]

(ii) About a Diameter. Here, again, due to symmetry, the M.I. of the disc, about one diameter is the same as about another. So that, if \( I \) be the M.I. of the disc about each of the perpendicular diameters \( XOX' \) and \( YOY' \), (Figure), we have, by the principle of perpendicular axes,
4.2.5 Moment of Inertia of an Annular Ring or Disc:

(i) **About an axis through its centre and perpendicular to its plane.** An annular disc is just an ordinary disc, with a smaller coaxial disc removed from it, leaving a concentric circular hole in it, as shown in figure.

If \( R \) and \( r \) be the outer and inner radii of the disc and its mass \( M \), we have

mass per unit area of the disc \( \frac{M}{\pi} \left( R^2 - r^2 \right) \).

Now, the disc may be imagined to be made up of a number of circular rings, with their radii ranging from \( r \) to \( R \). So that, considering one such ring of radius \( x \) and width \( dx \), we have

face area of the ring \( 2\pi x \, dx \) and

\[ \therefore \text{its mass} = 2\pi x \, dx \left[ \frac{M}{\pi} \left( R^2 - r^2 \right) \right] = \left[ \frac{2Mx}{(R^2 - r^2)} \right] \, dx \]

and hence its M.I. about the axis through \( O \) and perpendicular to its plane

\[ \frac{2Mx}{(R^2 - r^2)} \, dx \cdot x^2 = \frac{2Mx^2}{(R^2 - r^2)} \, dx \]

The M.I. of the whole annular disc i.e. \( I \), is therefore, given by the integral of the above expression between the limits \( x = r \) and \( x = R \)

\[ I = \int_{r}^{R} \frac{2Mx^3}{(R^2 - r^2)} \, dx = \frac{2M}{(R^2 - r^2)} \int_{r}^{R} x^3 \, dx = \frac{2M}{(R^2 - r^2)} \left[ \frac{x^4}{4} \right]_{r}^{R} \]

\[ = \frac{2M}{(R^2 - r^2)} \left( \frac{R^4 - r^4}{4} \right) = \frac{M \left( R^2 + r^2 \right)}{2} \]
(ii) About a Diameter. Due to symmetry, the M.I. of the annular disc about one diameter is the same as about another, say \( I \). Then, clearly in accordance with the principle of perpendicular axes, the sum of its moment of inertia about two perpendicular diameters must be equal to its M.I. about the axis passing through its centre (where the two diameters intersect) and perpendicular to its plane,

\[
i + I = \frac{M(R^2 + r^2)}{2} \quad \text{or} \quad 2I = \frac{M(R^2 + r^2)}{2}, \quad \text{whence,} \quad I = \frac{M(R^2 + r^2)}{4}.
\]

4.2.6 Moment of Inertia of a Solid Cylinder:

(i) About its own axis of cylindrical symmetry. A solid cylinder is just a thick circular disc or a number of thin circular disc (all of the same radius) piled up one over the other, so that its axis of cylindrical symmetry is the same as the axis passing through the centre of the thick disc (or the pile of thin discs) and perpendicular to its plane.


∴ M.I. of the thick disc (or the pile of thin discs) of the same mass and radius about the axis through its centre and perpendicular to its plane,

or

\[I = \frac{MR^2}{2}.
\]

(ii) About the axis through its centre and perpendicular to its axis of cylindrical symmetry. If \( R \) be the radius, \( l \), the length and \( M \), the mass of the solid cylinder, supposed to be uniform and of a homogeneous composition, we have its mass per unit length = \( \frac{M}{l} \).

Now, imagining the cylinder to be made up of a number of discs each of radius \( R \), placed adjacent to each other, and considering one such disc of thickness \( dx \) and at a distance \( x \) from the centre \( O \) of the cylinder, (figure), we have

\[
\text{Mass of the disc} = \left( \frac{M}{l} \right)dx \quad \text{and radius} = R
\]

And \( \therefore \) M.I. of the disc about its diameter \( AB = \frac{M}{l}dx \cdot \frac{R^2}{4} \) and its M.I. about the parallel axis \( YOY' \), passing through the centre \( O \) of the cylinder and perpendicular to its axis of cylindrical symmetry (or its length), in accordance with the principle of parallel axes,
Hence, M.I. of the whole cylinder about this axis, i.e. \( I = \) twice the integral of the above expression between the limits \( x = 0 \) and \( x = \frac{l}{2} \),

\[
I = 2\int_0^{l/2} \left( \frac{M}{l} \cdot \frac{R^2}{4} dx + \frac{M}{l} x^2 dx \right) = \frac{2M}{l} \int_0^{l/2} \left( \frac{R^2}{4} dx + x^2 dx \right)
\]

\[
= \frac{2M}{l} \left[ \frac{R^2 x}{4} + \frac{x^3}{3} \right]_0^{l/2}
\]

or

\[
I = \frac{2M}{l} \left[ \frac{R^2 \cdot l}{4} + \frac{l^3}{3} \right] - \frac{2M}{l} \left( \frac{R^2 l}{8} + \frac{l^3}{24} \right) = M \left( \frac{R^2}{4} + \frac{l^2}{12} \right)
\]

### 4.2.7 Moment of Inertia of a Solid Cone:

(i) **About its vertical axis.** Let \( M \) be the mass of the solid cone, \( h \) its vertical height and \( R \), the radius of its base (figure below).

Clearly, volume of the cone \( = \frac{1}{3} \pi R^2 h \) and if \( \rho \) be the density of its material its mass \( M = \frac{1}{3} \pi R^2 h \rho \), whence, \( \rho = \frac{3M}{\pi R^2 h} \).

Now, the cone may be imagined to consist of a number of discs of progressively decreasing radii, from \( R \) to 0 piled up one over the other.

Considering one such disc of thickness \( dx \) and at a distance \( x \) from the vertex \( A \) of the cone, we have

Radius of the disc, \( r = x \tan \alpha \),
where $\alpha$ is the semi-vertical angle of the cone, and, therefore its volume

$$\pi r^2 dx = \pi x^2 \tan^2 \alpha dx$$

and its mass $\pi x^2 \tan^2 \alpha \rho dx$.

Hence, M.I. of the disc about the vertical axis $AO$ of the cone (i.e. an axis passing through its centre and perpendicular to its plane) = mass $\times \left(\frac{\text{radius}}{2}\right)^2$

$$= \pi x^2 \rho \tan^2 \alpha dx \cdot \frac{r^2}{2} = \frac{\pi x^2 \rho \tan^2 \alpha dx \cdot x^2 \tan^2 \alpha}{2} = \left(\frac{\pi \rho \tan^4 \alpha}{2}\right)x^4dx$$

And $\because$ M.I. of the entire cone about its vertical axis $AO$ is given by

$$I = \int_0^h \frac{\pi \rho \tan^4 \alpha}{2} x^4 dx = \frac{\pi \rho \tan^4 \alpha}{2} \int_0^h x^4 dx = \frac{\pi \rho \tan^4 \alpha}{2} \left[\frac{x^5}{5}\right]_0^h$$

$$= \frac{\pi \rho \tan^4 \alpha h^5}{10} = \frac{\pi \rho R^4 h^5}{20}$$

substituting $R/h$ for $\tan \alpha$

or, substituting the value of $\rho$ obtained above, we have

$$I = \frac{\pi \cdot 3M}{\pi R^2 h} \cdot \frac{R^4 h^5}{20} = \frac{3MR^2 h^5}{10}$$

(ii) About an axis passing through the vertex and parallel to its base.

Again, considering the disc at a distance $x$ from the vertex of the cone, we have its M.I. about its diameter $\text{mass} \times \left(\frac{\text{radius}}{2}\right)^2$

$$= \pi x^2 \rho \tan^2 \alpha dx \cdot \frac{r^2}{4} = \frac{\pi x^2 \rho \tan^2 \alpha dx \cdot x^2 \tan^2 \alpha}{4} = \left(\frac{\pi \rho \tan^4 \alpha}{4}\right)x^4dx$$

$\because$ Its M.I. about the parallel axis $XX'$, parallel to its base is given by

$$= \left(\frac{\pi \rho \tan^4 \alpha}{4}\right)x^4 dx + \pi x^2 \tan^2 \alpha \rho x^2 dx = \left(\frac{\pi \rho \tan^4 \alpha}{4}\right)x^4 dx + \pi \rho \tan^2 \alpha x^4 dx$$

Hence M.I. of the entire cone about the axis $XX'$, parallel to the base is given by

$$I = \int_0^h \frac{\pi \rho \tan^4 \alpha}{4} x^4 dx + \pi \rho \tan^2 \alpha x^4 dx = \frac{\pi \rho \tan^4 \alpha}{4} \int_0^h x^4 dx + \pi \rho \tan^2 \alpha \int_0^h x^4 dx$$
4.2.8 Moment of Inertia of a Hollow Cylinder

(i) About its axis of cylindrical symmetry. A hollow cylinder may be considered to be a thick annular disc or a combination of thin annular discs, each of the same external and internal radii, placed adjacent to each other, the axis of the cylinder (i.e. its axis of cylindrical symmetry) being the same as the axis passing through the centre of the thick annular disc (or the combination of thin annular discs) and perpendicular to its plane.

The M.I. of the hollow cylinder about its own axis is, therefore, the same as that of a thick annular disc (or a combination of thin annular discs) of the same mass $M$ and external and internal radii $R$ and $r$ respectively about the axis passing through its centre and perpendicular to its plane.

\[ I = \frac{M}{2} \left( R^2 + r^2 \right) \]

i.e.,

\[ I = \frac{M}{2} \left( R^2 + r^2 \right) \]

Alternatively, we may obtain the same result directly as follows:

Let $R$ and $r$ be the external and internal radii respectively of a hollow cylinder of length $l$ and mass $M$, (figure below). Then face-area of the cylinder \( = \pi \left( R^2 - r^2 \right) \) and its volume \( = \pi \left( R^2 - r^2 \right) l \) and, therefore, its mass per unit volume \( = \frac{M}{\pi \left( R^2 - r^2 \right) l} \).
Imagining the cylinder to be made up of a large number of thin, coaxial cylinders, with their radii varying from \( r \) to \( R \) and considering one such cylinder of radius \( x \) and thickness \( dx \), we have its face area = \( 2\pi xdx \) and its volume = \( 2\pi xdxl \) and hence its mass = \( \frac{2\pi xdx \cdot l \times M}{\pi (R^2 - r^2)} = \frac{2Mxdx}{(R^2 - r^2)} \).

And therefore, its M.I. about the axis of the cylinder

\[
= \frac{2Mxdx}{(R^2 - r^2)} x^2 = \frac{2M}{(R^2 - r^2)} x^3 dx
\]

Hence, M.I. of the entire cylinder about its axis, i.e. \( I \), is given by

\[
I = \int_{r}^{R} \frac{2M}{(R^2 - r^2)} x^2 dx = \frac{2M}{(R^2 - r^2)} \int_{r}^{R} x^3 dx = \frac{2M}{(R^2 - r^2)} \left[ \frac{X^4}{4} \right]_{r}^{R},
\]

\[
= \frac{2M}{(R^2 - r^2)} \left( \frac{R^4 - r^4}{4} \right) = M \left( \frac{R^2 + r^2}{2} \right)
\]

(ii) About an axis passing through its centre and perpendicular to its own axis. Again, if \( R \) and \( r \) be the external and internal radii respectively of the hollow cylinder, \( I \), its length and \( M \), its mass, we have mass per unit volume of the cylinder = \( \frac{M}{\pi (R^2 - r^2) l} \).

Imagining the hollow cylinder to be made up of a large number of annular discs, of external and internal radii \( R \) and \( r \) respectively, placed adjacent to each other, and considering one such disc at a distance \( x \) from the axis \( YOY' \) passing through the centre \( O \) of the cylinder and perpendicular to its own axis, we have

surface area of the disc = \( \pi (R^2 - r^2) \), volume

\( = (R^2 - r^2) dx \) and therefore, its mass

\[
= \pi (R^2 - r^2) dx \times \frac{M}{\pi (R^2 - r^2) l} = \frac{Mdx}{l}
\]
So that, M.I. of the disc about its diameter \((AB) = \frac{Mdx}{l} \cdot \frac{(R^2 + r^2)}{4}\)

Hence, M.I. of entire hollow cylinder about the axis \(YOY'\) is equal to twice the integral of the above expression between the limits \(x = 0\) and \(x' = \frac{l}{2}\), i.e.

\[
I = 2 \int_{0}^{l/2} \left[ \frac{M(R^2 + r^2)}{4l} dx + \frac{M}{l} x^2 dx \right] = \frac{2M}{l} \int_{0}^{l/2} \left[ \frac{R^2 + r^2}{4} dx + x^2 dx \right]
\]

\[
= \frac{2M}{l} \left[ \frac{(R^2 + r^2)x}{4} + \frac{x^3}{3} \right]_{0}^{l/2}
\]

or

\[
I = \frac{2M}{l} \left[ \frac{(R^2 + r^2)l}{4 \times 2} + \frac{l^3}{8 \times 3} \right] = M \left[ \frac{(R^2 + r^2)}{4} + \frac{l^2}{12} \right]
\]

### 4.2.9 Moment of Inertia of a Uniform Hollow Sphere about a Diameter

Let \(M\) and \(R\) be the mass and the radius of the sphere \(O\) its centre and \(OX\) the given axis. The mass is spread over the surface of the sphere and the inside is hollow.

Let us consider a radius \(OA\) of the sphere at an angle \(\theta\) with the axis \(OX\) and rotate this radius about \(OX\). The point \(A\) traces a circle on the sphere. Now change \(\theta\) to \(\theta + d\theta\) and get another circle of somewhat larger radius on the sphere. The part of the sphere between these two circles, shown in the figure, forms a ring of radius \(R\sin \theta\). The width of this ring is \(R\) and its periphery is \(2\pi R \sin \theta\). Hence,

the area of the ring \(= (2\pi R \sin \theta)(R \sin \theta)\).

Mass per unit area of the sphere \(= \frac{M}{4\pi R^2}\)

The mass of the ring \(= \frac{M}{4\pi R^2}(2\pi R \sin \theta)(R \sin \theta) = \frac{M}{2} \sin \theta d\theta\)

The moment of inertia of this elemental ring about \(OX\) is
As $\theta$ increases from 0 to $\pi$, the elemental rings cover the whole spherical surface. The moment of inertia of the hollow sphere is, therefore,

$$
I = \int_{0}^{\pi} \frac{2}{R^2} \left[ \frac{1}{2} \int_{0}^{\theta} \sin^3 \theta \sin \theta \, d\theta \right] \, d\theta
$$

$$
= \frac{2}{R^2} \left[ \frac{1}{3} \cos^3 \theta \right]_{\theta=0}^{\theta} = \frac{2}{3} MR^2.
$$

### 4.2.10 Moment of Inertia of a Uniform Solid Sphere about a Diameter

Let $M$ and $R$ be the mass and radius of the given solid sphere. Let $O$ be the centre and $OX$ the given axis. Draw two spheres of radii $x$ and $x+dx$ concentric with the given solid sphere. The thin spherical shell trapped between these spheres may be treated as a hollow sphere of radius $x$.

The mass per unit volume of the solid sphere

$$
= \frac{M}{4\pi R^3} = \frac{3M}{4\pi R^3}
$$

The thin hollow sphere considered above has a surface area $4\pi x^2$ and thickness $dx$. Its volume is $4\pi x^2 dx$ and hence its mass is

$$
= \left( \frac{3M}{4\pi R^3} \right) (4\pi x^2 \, dx) = \frac{3M}{R^3} x^2 \, dx
$$

Its moment of inertia about the diameter $OX$ is, therefore,

$$
dI = \frac{2}{3} \left[ \frac{3M}{R} x^2 \, dx \right] x^2 = \frac{2M}{R^3} x^4 \, dx
$$

If $x=0$, the shell is formed at the centre of the solid sphere. As $x$ increases from 0 to $R$, cover the whole solid sphere.

The moment of inertia of the solid sphere about $OX$ is, therefore,

$$
I = \int_{0}^{R} \frac{2M}{R^3} x^4 \, dx = \frac{2}{5} MR^2.
$$
4.3 Rotational Dynamics

4.3.1 Angular Momentum

Consider motion in the $xy$ plane, first in the $x$ direction and then in the $y$ direction, as in drawings $a$ and $b$ on the next page.

The most general case involves both these motions simultaneously, as drawings above show. Hence $L_z = xp_y - yp_x$ as you can verify by inspection or by evaluating the cross product as follows. Using $r = (x, y, 0)$ and $p = (p_x, p_y, 0)$, we have

$$L = r \times p = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & 0 \\ p_x & p_y & 0 \end{vmatrix} = (xp_y - yp_x)\hat{k}$$

We have limited our illustrations to motion in the $xy$ plane where the angular momentum lies entirely along the $z$ axis. There is however, no difficulty applying any of these methods to the general case where $L$ components along all three axes.
4.3.2 Torque

The torque \( \vec{\tau} = \vec{r} \times \vec{F} \) that \( \vec{\tau} \) and \( \vec{F} \) are always perpendicular. There can be a torque on a system with zero net force, and there can be force with zero net torque. In general, there will be both torque and force. These three cases are illustrated in the sketches below. (The torques are evaluated about the centers of the disks.)

Torque is important because it is intimately related to the rate of change of angular momentum:

\[
\frac{d\vec{L}}{dt} = \frac{d}{dt} (\vec{r} \times \vec{p}) = \left( \frac{d\vec{r}}{dt} \times \vec{p} \right) + \left( \vec{r} \times \frac{d\vec{p}}{dt} \right)
\]

But \( (d\vec{r}/dt) \times \vec{p} = \vec{v} \times mv = 0 \), since the cross product of two parallel vectors is zero. Also, \( d\vec{p}/dt = \vec{F} \), by Newton’s second law. Hence, the second terms is \( \vec{r} \times \vec{F} = \vec{\tau} \), and we have

\[
\vec{\tau} = \frac{d\vec{L}}{dt}
\]

shows that if the torque is zero, \( \vec{L} \) = constant and the angular momentum is conserved.

**Example:** Consider a block of mass \( m \) and negligible dimensions sliding freely in the \( x \) direction with velocity \( \vec{v} = v_1 \hat{i} \) as shown in the sketch.

(a) What is its angular momentum \( L_A \) about origin \( A \) and its angular momentum \( L_B \) about the origin \( B \)?

(b) What is its torque \( L_A \) about origin \( A \) and its angular momentum \( L_B \) about the origin \( B \)?
Solution: As shown in the figure given below the vector from origin $A$ to the block is $r_A = x\hat{i}$. Since $r_A = mr_A \times v = 0$

Taking origin $B$, we can resolve the position vector $r_B$ into a component $r_\parallel$ parallel to $v$ and a component $r_\perp$ perpendicular to $v$. Since $r_\parallel \times v = 0$, only $r_\perp$ gives a contribution to $L_B$. We have $|r_\perp \times v| = lv$ and

$L_B = mr_B \times v = ml\hat{k}$

$L_B$ lies in the positive $z$ direction because the sense of rotation is counter-clockwise about the $z$-axis.

To calculate $L_B$ formally we can write $r_B = xi - yj$ and evaluate $r_B \times v$ using our determinantal form.

\[
L_B = mr_B \times v = m \begin{vmatrix} i & j & k \\ v & 0 & 0 \end{vmatrix} = ml\hat{k}
\]

The following example shows in a striking way how $L$ depends on our choice of origin.

b) For a simple illustration of the relation $\tau = dL/dt$, consider a small block of mass $m$ sliding in the $z$ direction with velocity $v = v\hat{i}$. The angular momentum of the block about origin $B$ is

$L_B = mr_B \times v = ml\hat{k} \quad \text{(1)}$
as we discussed in problem (a). If the block is sliding freely, $v$ does not change, and $L_B$ is therefore constant, as we expect, since there is no torque acting on the block.

Suppose now that the block slows down because of a friction force $f = -f \hat{i}$. The torque on the block about origin $B$ is

$$\tau_B = r_B \times f = -lf \hat{k}$$

(2)

We see from Eq. (1) that as the block slows, $L_B$ remains along the positive $z$ direction but its magnitude decreases. Therefore, the change $\Delta L_B$ in $L_B$ points in the negative $z$ direction, as shown in the lower sketch. The direction of $\Delta L_B$ is the same as the direction of $\tau_B$. Since $\tau = dL / dt$ in general, the vectors $\tau$ and $\Delta L$ are always parallel.

From Eq. (1), $\Delta L_B = ml\Delta v \hat{k}$

(3)

Where $\Delta v < 0$. Dividing equation (3) by $\Delta t$ and taking the limit $\Delta t \to 0$, we have

$$\frac{dL_B}{dt} = ml \frac{dv}{dt} \hat{k}$$

(4)

By Newton’s second law, $m dv / dt = -f$ and equation (4) becomes

$$\frac{dL_B}{dt} = -lf \hat{k} = \tau_B$$, as we expect.

It is important to keep in mind that since $\tau$ and $L$ depend on the choice of origin, the same origin must be used for both when applying the relation $\tau = dL / dt$, as we were careful to do in this problem. The angular momentum of the block in this example changed only in magnitude and not in direction, since $\tau$ and $L$ happened to be along the same line.
**Example:** Assume that the Conical pendulum is in steady circular motion with constant angular velocity $\omega$, then discuss angular momentum and Torque, for the Conical Pendulum about point $A$ and point $B$

**Solution:** To calculate of angular momentum

We begin by evaluating $L_A$, the angular momentum about origin $A$.

From the sketch we see that $L_A$ lies in the positive $z$ direction. It has magnitude $|r| \times |p| = |r||p| = rp$, where $r$ is the radius of the circular motion. Since $|p| = Mv = Mr\omega$ we have $L_A = Mr^2\omega k$

Note that $L_A$ is constant, both in magnitude and direction.

Now let us evaluate the angular momentum about the origin $B$ located at the pivot. The magnitude of $L_B$ is $\left| L_B \right| = |r' \times p| = |r'| |p| = l|p| = Mr\omega$

where $|r'| = l$ the length of the string. It is apparent that the magnitude of $L$ depends on the origin we choose.

Unlike $L_A$, the direction of $L_B$ is not constant. $L_B$ is perpendicular to both $r'$ and $p$, and the sketches below show $L_B$ at different times.

Two sketches are given to emphasize that only the magnitude and direction of $L$ are important, not the position at which we choose to draw it. The magnitude of $L_B$ is constant, but its direction is obviously not constant as the bob swings around, $L_B$ sweeps out the shaded cone shown in the sketch at the right. The $z$ component of $L_B$ is constant, but the horizontal component travels around the circle with the bob.
Calculation of Torque

we calculated the torque of a conical pendulum about two different origins. Now we shall complete the analysis by showing that the relation $\tau = dL/dt$ is satisfied.

The sketch illustrates the forces on the bob. $T$ is the tension in the string. For uniform circular motion there is no vertical acceleration, and consequently
\[ T \cos \alpha - Mg = 0. \]  

(1)

The total force $F$ on the bob is radially inward: $F = -T \sin \alpha \hat{r}$.

The torque on $M$ about $A$ is $\tau_A = r_A \times F = 0$,

since $r_A$ and $F$ are both in the $\hat{r}$ direction. Hence $\frac{dL_A}{dt} = 0$, and we have the result $L_A = \text{constant}$, as we already know.

The problem looks entirely different if we take the origin at $B$. The torque $\tau_B$ is $\tau_B = r_B \times F$.

Hence, $|\tau_B| = l \cos \alpha F = l \cos \alpha F = l \cos \alpha T \sin \theta = Mgl \sin \alpha$

where we have used Eq. (1). $T \cos \alpha = Mg$. The direction of $\tau_B$ is tangential to the line of motion of $M$.

$\tau_B = Mgl \sin \alpha \hat{\theta}$

where $\hat{\theta}$ is the unit tangential vector in the plane of motion.

Our problem is to show that the relation
\[ \tau = \frac{dL_B}{dt} \]

is satisfied. we know that $L_B$ has constant magnitude $Ml \omega$. As the diagram at left shows, $L_B$ has a vertical component $L_z = Ml \omega \sin \alpha$ and a horizontal radial component $L_r = Ml \omega \cos \alpha$. Writing $L_B = \tilde{L}_z + \tilde{L}_r$, we see that $L_z$ is constant; it changes direction as the bob swings around. However, the magnitude of $L_r$ is constant. We encountered such a situation in section 1.8, where we showed that the only way a vector $A$ of constant
magnitude can change in time is to rotate and that if its instantaneous rate of rotation is \( \frac{d\theta}{dt} \), then \( \frac{dA}{dt} = A \frac{d\theta}{dt} \). We can employ this relation directly to obtain
\[
\left| \frac{dL}{dt} \right| = L_r \omega.
\]

However, since we shall invoke this result frequently, let us take a moment to rederive it geometrically.

The vector diagrams show \( L \) at some time \( t \) and at \( t + \Delta t \) During the interval \( \Delta t \), the bob swings through angle \( \Delta \theta = \omega \Delta t \), and \( L \), rotate through the same angle. The magnitude of the vector difference \( \Delta L_r = L_r (t + \Delta t) - L_r (t) \) is given approximately by
\[
|\Delta L_r| \approx L_r \Delta \theta
\]

In the limit \( \Delta t \to 0 \), we have
\[
\frac{dL_r}{dt} = L_r \frac{d\theta}{dt} = L_r \omega
\]

Since \( L_r = Mr \omega \cos \alpha \), \( \frac{dL_r}{dt} = Mr \omega^2 \cos \alpha \).

\( Mr \omega^2 \) is the radial force, \( T \sin \alpha = Mg \), we have \( \frac{dL_r}{dt} = Mgl \sin \alpha \)

which agrees with the magnitude of \( \tau_{\parallel} \) from Eq. (2), Furthermore, as the vector drawings indicate, \( \frac{dL_r}{dt} \) lies in the tangential direction. Parallel to \( \tau_{\parallel} \), as we expect.

Another way to calculate \( \frac{dL_{\parallel}}{dt} \) is to write \( L_{\parallel} \) in vector form and then differentiate:
\[
\ddot{L}_{\parallel} = (Mr \omega \sin \alpha) \hat{k} + (Mr \omega \cos \alpha) \hat{\dot{r}}
\]
\[
\frac{dL_{\parallel}}{dt} = Mr \omega \cos \alpha \frac{d\hat{r}}{dt} = Mr \omega^2 \cos \alpha \dot{\theta} \quad \text{where we have used} \quad \frac{d\hat{r}}{dt} = \omega \dot{\theta}.
\]
Example: Atwood’s Machine with a Massive Pulley

The problem is to find the acceleration $a$ for the arrangement shown in the sketch. The effect of the pulley is to be included. The masses of particles are $M_1, M_2$ and mass of pulley is $M_p$. The radius of pulley $R$.

**Solution:** Force diagrams for the three masses are shown below. The points of application of the forces on the pulley are shown; this is necessary whenever we need to calculate torques. The pulley evidently goes pure rotation about its axis, so we take the axis of rotation to be the axle.

The equations of motion are

\[
W_1 - T_1 = M_1 a \\
T_2 - W_2 = M_2 a \]

Masses

\[
\gamma = T_1 R - T_2 R = I \alpha \\
N - T_1 - T_2 - W_p = 0
\]

Pulley

Note that in the torque equation, $a$ must be positive counterclockwise to correspond to our convention that torque out of the paper is positive. $N$ is the force on the axle, and the last equation simply assures that the pulley does not fall. Since we don’t need to know $N$ it does not contribute to the solution.

There is a constraint relating $a$ and $\alpha$, assuming that the rope does not slip. The velocity of the rope is the velocity of a point on the surface of the wheel, $v = \omega R$, from which it follows that $a = \alpha R$. 
We can now eliminate \( T_1, T_2 \) and \( \alpha \):

\[
W_1 - W_2 - (T_1 - T_2) = (M_1 + M_2)a
\]

\[
T_1 - T_2 = \frac{I\alpha}{R} = \frac{Ia}{R^2}
\]

\[
W_1 - W_2 - \frac{Ia}{R^2} = (M_1 + M_2)a
\]

If the pulley is a simple disk, we have

\[
I = \frac{M_r R^2}{2}
\]

and it follows that

\[
a = \frac{(M_1 - M_2)g}{M_1 + M_2 + \frac{M_r}{2}}.
\]

The Pulley increases the total inertial mass of the system, but in comparison with the hanging weights, the effective mass of the pulley is only one-half its real mass.

4.4 Motion Involving Both Translation and Rotation

By using center of mass coordinates we will find it a straightforward matter to obtain simple expressions for both the angular momentum and the torque and to find the dynamical equation connecting them.

As before, we shall consider only motion for which the axis of rotation remains parallel to the \( z \)-axis. We shall show that \( L_z \), the \( z \) component of the angular momentum of the body, can be written as the sum of two terms. \( L_z \) is the angular momentum \( I_0 \omega \) due to rotation of the body about its center of mass, plus the angular momentum \( (R \times MV)_z \) due to motion of the center of mass with respect to the origin of the inertial coordinate system:

\[
L_z = I_0 \omega + (R \times MV)_z
\]

where \( R \) is the position vector of the center of mass and \( V = \dot{R} \).
### 4.4.1 Angular Momentum of a Rolling Wheel

In this example we apply to the calculation of the angular momentum of a uniform wheel of mass $M$ and radius $b$ which rolls uniformly and without slipping. The moment of inertia of the wheel about its center of mass is $I_0 = \frac{1}{2} Mb^2$ and its angular momentum about the center of mass is

$$L_0 = -I_0 \omega = -\frac{1}{2} Mb^2 \omega .$$

$L_0$ is parallel to the axis. The minus sign indicates that $L_0$ is directed into the paper in the negative $z$ direction.

If we calculate the angular momentum of the center of mass of the wheel with respect to the origin, we have

$$\left( R \times MV \right)_z = -MbV$$

The total angular momentum about the origin is then

$$L_z = -\frac{1}{2} Mb^2 \omega - MbV = -\frac{1}{2} Mb^2 \omega = -\frac{1}{2} Mb^2 \omega$$

where we have used the result $V = b \omega$, which holds for a wheel that rolls without slipping.

Torque also naturally divides itself into two components.

### 4.4.2 The Torque on a Body

Fixed axis rotation $\vec{\omega} = \hat{z} \omega$.

If $\tau_o$ is the component of the torque about the center of mass and $\mathbf{F} = \sum f_j$ is the total applied force

Then torque is given by $\tau_z = \tau_o + \left( \mathbf{R} \times \mathbf{F} \right)_z$

The first term in $\tau_z$ is the torque about the center of mass due to the various external forces, and the second term is the torque due to the total external force acting at the center of mass.
Example: A disk of mass $M$ and radius $b$ is pulled with constant force $F$ by a thin tape wound around its circumference. The disk slides on ice without friction. What will be torque and acceleration of the system?

We choose a coordinate system whose origin $A$ is along the line of $F$. The torque about $A$ is, from

$$\tau_z = \tau_0 + (R \times F)_z = bF - bF = 0$$

The torque is zero. As we expect, and angular momentum about the origin is conserved. The angular momentum about $A$ is,

$$L_z = I_0 \omega + (R \times MV)_z = I_0 \omega - bMV.$$  

Since $\frac{dL_z}{dt} = 0$, we have

$$0 = I_0 \alpha - bMa$$

or

$$\alpha = \frac{bMa}{I_0} = \frac{bF}{I_0},$$

as before.

4.4.3 Condition of Rolling Without Slipping

If velocity of center of mass is $V$ and $\omega$ is angular velocity of center of mass then $V = \omega R$ where $R$ is radius of rolling body. Similarly if acceleration of center of mass is $a$ and $\alpha$ is angular velocity of about center of mass then $a = \alpha R$ where $R$ is radius of rolling body.

Example - Drum Rolling Down a Plane

A uniform drum of radius $b$ and mass $M$ rolls a without slipping down a plane inclined at angle $\theta$. Find its velocity along the plane. The moment of inertia of the drum about its axis is

$$I_0 = \frac{Mb^2}{2}.$$  

Then find the speed at the lowest end of inclined plane

where $l = \frac{h}{\sin \theta}$ is the displacement of the center of mass as the drum descends height $h$.

Change in potential energy is equal to gain in kinetic energy

So $Mgh = \frac{1}{2} (M)V^2 + \frac{1}{2} (I_0)\omega^2$
Example: A uniform drum of radius \( b \) and mass \( M \) rolls a without slipping down a plane inclined at angle \( \theta \). Find its acceleration along the plane. The moment of inertia of the drum about its axis is \( I_0 = \frac{Mb^2}{2} \).

Solution: The forces acting on the drum are shown in the diagram. \( f \) is the force of friction. The translation of the center of mass along the plane is given by

\[
W \sin \theta - f = Ma
\]

and the rotation about the center of mass by

\[
b \theta = I_0 \alpha.
\]

For rolling without slipping, we also have

\[a = b \alpha \,.
\]

If we eliminate \( f \), we obtain

\[
W \sin \theta - I_0 \frac{\alpha}{b} = Ma
\]

Using \( I_0 = \frac{Mb^2}{2} \), and \( \alpha = \frac{a}{b} \), we obtain

\[
Mg \sin \theta - \frac{Ma}{2} = Ma,
\]

or

\[a = \frac{2}{3} g \sin \theta.\]
Example: A stick of length \( l \) and mass \( M \) initially upright on a frictionless table, starts falling. The problem is to find the speed of the center of mass as a function of position. The key lies in realizing that since there are no horizontal forces, the center of mass must fall straight down. Since we must find velocity as a function of position, it is natural to apply energy methods.

The sketch shows the stick after it has rotated through angle \( \theta \) and the center of mass has fallen distance \( y \). The initial energy is \( E = K_0 + U_0 = \frac{Mgl}{2} \).

The kinetic energy at a later time is \( K = \frac{1}{2} I_0 \dot{\theta}^2 + \frac{1}{2} M \dot{y}^2 \) and the corresponding potential energy is \( U = Mg \left( \frac{l}{2} - y \right) \).

Since there are no dissipative forces, mechanical energy is conserved and \( K + U = K_0 + U_0 = \frac{Mgl}{2} \). Hence

\[
\frac{1}{2} M \dot{y}^2 + \frac{1}{2} I_0 \dot{\theta}^2 + Mg \left( \frac{l}{2} - y \right) = Mg \frac{l}{2}.
\]

We can eliminate \( \theta \) by turning to the constraint equation. From the sketch we see that

\[ y = \frac{l}{2} (1 - \cos \theta). \]

Hence, \( \dot{y} = \frac{l}{2} \sin \theta \dot{\theta} \) and \( \dot{\theta} = \frac{2}{l \sin \theta} \dot{y} \).

Since \( I_0 = M \left( \frac{l^2}{12} \right) \), we obtain

\[
\frac{1}{2} M \dot{y}^2 + \frac{1}{2} M \frac{l^2}{12} \left( \frac{2}{l \sin \theta} \right) \dot{\theta}^2 + Mg \left( \frac{l}{2} - y \right) = Mg \frac{l}{2}.
\]

or

\[
\dot{y}^2 = \frac{2gy}{1} \left[ \frac{1}{1 + \frac{1}{3 \sin^2 \theta}} \right] \Rightarrow \dot{y} = \left[ \frac{6gy \sin^2 \theta}{3 \sin^2 \theta + 1} \right]^{1/2}
\]
MCQ (Multiple Choice Questions)

Q1. A thin wire of length $L$ and uniform linear mass density $\rho$ is bent into a circular loop with centre of $O$ as shown. The moment of inertia of the loop about the axis $XX'$ is

(a) $\frac{\rho L^3}{8\pi^2}$  
(b) $\frac{\rho L^3}{16\pi^2}$  
(c) $\frac{5\rho L^3}{16\pi^2}$  
(d) $\frac{3\rho L^3}{8\pi^2}$

Q2. One quarter sector is cut from a uniform circular disc of radius $R$. This sector has a mass $M$. If is made to rotate about a line perpendicular to its plane and passing through the centre of the original disc. Its moment of inertia about the axis of rotational is

(a) $\frac{1}{2} MR^2$  
(b) $\frac{1}{4} MR^2$  
(c) $\frac{1}{8} MR^2$  
(d) $\sqrt{2} MR^2$

Q3. From a circular disc of radius $R$ and mass $9M$ a small disc of radius $\frac{R}{3}$ is removed from the disc. The moment of inertia of the remaining disc about an axis perpendicular to the plane of the disc and passing through $O$ is

(a) $4MR^2$  
(b) $\frac{40}{9} MR^2$  
(c) $10MR^2$  
(d) $\frac{37}{9} MR^2$

Q4. Four spheres of diameter $2a$ and mass $M$ each are placed with their centres on the four corners of a square of side $b$. The moment of inertia of the system about one side of the square taken as the axis is

(a) $\frac{8M(3a^2 + 2b^2)}{3}$  
(b) $\frac{9M(6a^2 + 4b^2)}{8}$  
(c) $\frac{3M(2a^2 + 4b^2)}{7}$  
(d) $\frac{2M(4a^2 + 5b^2)}{5}$

Q5. The ratio of radius of gyration of a circular ring and a disc of the same radius about the axis passing through their centres and perpendicular to their plane

(a) $\sqrt{5} : 1$  
(b) $\sqrt{2} : 1$  
(c) $1 : \sqrt{3}$  
(d) $1 : \sqrt{2}$
Q6. The moment of inertia of a thin rod of length \( L \) and \( M \) about an axis that is perpendicular to the rod and of a distance \( x \) from its centre is

(a) \( \frac{ML^2}{12} + Mx^2 \)  
(b) \( \frac{ML^2}{6} + Mx^2 \)  
(c) \( \frac{ML^2}{3} + \frac{Mx^2}{7} \)  
(d) \( \frac{ML^2}{12} + Mx^2 \)

Q7. Two uniform identical rods each of mass \( M \) and length \( l \) are joined to form a cross as shown in the figure. The moment of inertia of the cross about the bisector \( AB \) is

(a) \( \frac{Ml^2}{4} \)  
(b) \( \frac{Ml^2}{5} \)  
(c) \( \frac{Ml^2}{8} \)  
(d) \( \frac{Ml^2}{12} \)

Q8. The moment of inertia of a pair spheres, each having a mass \( m \) and radius \( r \), kept in contact about the tangent passing through the point of contact is

(a) \( \frac{7mr^2}{5} \)  
(b) \( \frac{9mr^2}{5} \)  
(c) \( \frac{14mr^2}{5} \)  
(d) \( \frac{18mr^2}{5} \)

Q9. From a given sample of uniform wire two circular loops \( P \) and \( Q \) are made, \( P \) of radius \( r \) and \( Q \) of radius \( nr \). If the M.I. of \( Q \) about its axis is four times that of \( P \) about its axis, the value of \( n \) is

(a) \( \frac{2}{3} \)  
(b) \( \frac{1}{3} \)  
(c) \( \frac{1}{4} \)  
(d) \( \frac{1}{4} \)

Q10. Two circular discs \( A \) and \( B \) are of equal masses and thickness but made of metal with densities \( d_A \) and \( d_B (d_A > d_B) \). If their moment of inertia about an axis passing through their centres and perpendicular to their faces are \( I_A \) and \( I_B \), then

(a) \( I_A = I_B \)  
(b) \( I_A > I_B \)  
(c) \( I_A < I_B \)  
(d) \( I_A \geq I_B \)
Q11. The angular momentum of a particle rotating under a central force is constant due to
   (a) constant force       (b) constant linear momentum
   (c) constant torque      (d) zero torque

Q12. A solid sphere is rotating in free space. If the radius of the sphere is increased keeping mass
     same which are of the following will not be affected:
   (a) moment of inertia     (b) angular momentum
   (c) angular velocity      (d) rotational kinetic energy

Q13. A thin horizontal circular disc is rotating about a vertical axis passing through its centre. An
     insect is at rest at a point near the rim of the disc. The insect now moves along a diameter of the
     disc to reach its outer end. During the journey of the insect, the angular speed of the disc
   (a) remains unchanged     (b) continuously decreases
   (c) continuously increases (d) first increases and then decreases

Q14. A couple is acting on a two-particle system connected by a massless rod. The resultant motion
     will be
   (a) purely rotational motion (b) purely translational motion
   (c) both (a) and (b)        (d) Neither (a) nor (b)

Q15. Angular momentum of a system of particles is conserved
   (a) when no external force acts on the system
   (b) when no external torque acts on the system
   (c) when no external impulse acts on the system
   (d) when axis of rotation remains the same

Q16. The direction of angular velocity vector is along
   (a) the tangent to the circular path (b) the inward radius
   (c) the outward radius            (d) the axis of rotation

Q17. A body is projected from the ground with some angle to the horizontal. What happens to its
     angular momentum about the initial position in its motion.
   (a) decreases             (b) increases
   (c) remains the same      (d) first increases then decreases
Q18. A disc is rolling, the velocity of its centre of mass is $v_{cm}$. Which one will be correct?

(a) the velocity of highest point is $2v_{cm}$ and the point of contact is zero.

(b) the velocity of highest point is $v_{cm}$ and the point of contact is $v_{cm}$.

(c) the velocity of highest point is $2v_{cm}$ and point of contact is $v_{cm}$.

(d) the velocity of highest point is $2v_{cm}$ and point of contact is $2v_{cm}$.

Q19. For a hollow cylinder and a solid cylinder rolling without slipping on an incline plane, which of them will reach earlier

(a) solid cylinder     (b) hollow cylinder

(c) both simultaneously    (d) can’t say anything

Q20. Pick out the correct statement about the torque vector

(a) The torque acting on a particle is given by $\vec{F} \times \vec{r}$ where $\vec{r}$ is the position vector of the particle about the point chosen for calculating torque.

(b) If net torque acting on a particle is zero, its angular momentum is not constant.

(c) Zero net torque acting on a system implies zero net force.

(d) If the angular momentum of a system changes with time, a net torque must act on the system.

Q21. A cylinder is released from rest from the top of an incline of inclination $\theta$ and length $l$. If the cylinder rolls without slipping, what will be its speed when it reaches the bottom?

(a) $\sqrt{\frac{4gl \sin \theta}{3}}$     (b) $\sqrt{\frac{2gl \sin \theta}{3}}$

(c) $\sqrt{\frac{5gl \sin \theta}{3}}$     (d) $\sqrt{\frac{6gl \sin \theta}{3}}$

Q22. A sphere of mass $M$ and radius $r$ shown in the figure slips on a rough horizontal surface. At some time it has translational velocity $v_0$ and rotational velocity about the centre of mass $\frac{v_0}{2r}$. When the sphere starts pure rolling the transitional velocity is

(a) $\frac{2v_0}{7}$     (b) $\frac{3v_0}{9}$     (c) $\frac{4v_0}{9}$     (d) $\frac{6v_0}{7}$
Q23. The centre of a wheel rolling on a plane surface moves with speed $v_0$. A particle on the rim of the wheel at the same level as the centre will be moving at speed:

(a) zero  (b) $v_0$  (c) $\sqrt{2}v_0$  (d) $2v_0$

Q24. Two small kids weighing 10 kg and 15 kg are trying to balance a seesaw of total length 5.0 m. If the kid of mass 10 kg sits at an end at what distance from the centre the other kid should sit?

(a) 1.2 m  (b) 1.5 m  (c) 1.3 m  (d) 1.7 m

Q25. A disc is performing pure rolling on a smooth stationary surface with constant angular velocity as shown in figure. At any instant for the lowermost point of the disc:

(a) velocity is $v$, acceleration is zero
(b) velocity is zero, acceleration is zero
(c) velocity is $v$, acceleration is $\frac{v^2}{R}$
(d) velocity is zero, acceleration is $\frac{v^2}{R}$
NAT (Numerical Answer type)

Q26. Three rings each of mass 10 kg and radius 2 m are placed such that they touch each other. The moment of inertia of about the axis shown is __________ kg \cdot m^2.

Q27. The moment of inertia of a solid cylinder of mass 10 kg and radius 3 m about a line parallel to axis of the cylinder and on the surface of the disc is ________ kg \cdot m^2.

Q28. A disc of mass 8 kg and radius 2 m is rotating about the axis AB, that is tangent to the disc. If the linear speed of point A on the periphery of the disc is 20 m/s, then the kinetic energy of the disc is __________ T.

Q29. Three point masses \( m_1 = 4 \text{ kg}, m_2 = 3 \text{ kg} \) and \( m_3 = 5 \text{ kg} \) are located at the vertices of an equilateral triangle of side length 5 m. The M.I. of the system about the median passing through mass \( m_1 \) is ________ kg \cdot m^2.

Q30. The M.I. of a uniform square plate of side 4 m and mass 6 kg about an axis perpendicular to its plane and passing through one of its corners is __________ kg \cdot m^2.

Q31. A disc of mass 2 kg and radius 1 m is rolling with angular speed 10 rad/s on a horizontal plane as shown in figure. The magnitude of angular momentum of the disc about the origin is ________.

Q32. A uniform body of radius 10 m, mass 5 kg and moment of inertia 2 kg \cdot m^2 rolls down an incline plane making an angle 37° with the horizontal. [Take \( \sin 37^\circ = 3/5 \), \( \cos 37^\circ = 4/5 \) and \( g = 10 \text{ m/s}^2 \)]. Then the acceleration is __________ m/s^2.

Q33. A solid sphere is rolling on a frictionless surface with a translational velocity \( v \) (in ms^{-1}) as shown in the figure. If it is to climb the incline plane, then \( v \) should be ________ m/s.

\[ h = 700 \text{ m} \]
Q34. A uniform cube of side 4 m and mass 30 kg rests on a rough horizontal plane. A horizontal force $F$ is applied normal to one of the faces at a point that is directly above the centre of the face at a height 3 m above the box. The minimum value of $F$ for which the cube begins to tip above the edge (assuming no sliding) is _________. (Take $g = 10 \text{m/s}^2$).

Q35. A stone of mass $m$, tied to the end of a spring, is whirled around a horizontal circle. The length of the string is gradually reduced keeping the angular momentum of the stone about the centre constant. Then the tension in the string is given by $T = Ar^n$, where $A$ is a constant, $r$, the instantaneous radius of the circle. Then $n$ is ________.

Q36. A rod of weight 20 N is supported by two parallel knife edges $A$ and $B$ and is in equilibrium in a horizontal position. The knives are at a distance 30 m from each other. The centre of mass of the rod is at a distance of 10 m from $A$. The normal reaction on $A$ is ___________ N.

Q37. A thin uniform circular disc of mass $M$ and radius $R$ is rotating in a horizontal plane about an axis passing through its centre and perpendicular to its plane with an angular velocity $10 \text{ rad/s}$. Another disc of same dimensions but of mass $\frac{M}{4}$ is placed gently on the first disc coaxially. The angular velocity of the system now is ________ rad/s.

Q38. A rod of length 1 m and mass 2 kg is hinged at point $O$. A small bullet of mass 1 kg hits the rod as shown in the figure. The bullet gets embedded in the rod. Angular velocity of the system just after impact, is ___________ rad/s².

Q39. A block of mass 2 kg is attached to the end of an inextensible string which is wound over a rough pulley of mass 6 kg. Assume the string does not slide over the pulley. The acceleration of the block when released is _______ m/s².
Q40. A block of mass $150 \text{ kg}$ is attached to a chord passing through a hole in a horizontal frictionless surface. The block is initially revolving in a circle of radius $0.5 \text{ m}$ about the hole, with tangential velocity of $4 \text{ m/s}$. The chord is then pulled slowly from below, shortening the radius of the circle in which the block revolves. The breaking strength of the chord is $600 \text{ N}$. The radius of the circle when the chord breaks is $__________ \text{ m}$.

**MSQ (Multiple Select Questions)**

Q41. The moment of inertia of a thin square plate $ABCD$ of uniform thickness about an axis passing through the centre $O$ and perpendicular to the plate is/are

(a) $I_1 + I_2$  
(b) $I_3 + I_4$  
(c) $I_1 + I_3$  
(d) $I_1 + I_2 + I_3 + I_4$  
(where $I_1, I_2, I_3, \text{ & } I_4$ are respectively the moment of inertia about axes 1, 2, 3 & 4)

Q42. Consider a disk of mass $M$ and radius $R$ and the axes shown in the figure.

(a) The moment of inertia of the disc about an axis passing through the centre and perpendicular to the disc is $\frac{MR^2}{2}$.

(b) The moment of inertia of the disc about the line $AB$ lying in the plane of the disc is $\frac{MR^2}{2}$.

(c) The moment of inertia of the disc about the line $CD$ is $\frac{5}{4}MR^2$.

(d) The moment of inertia of the disc about the line $CD$ is $\frac{3}{2}MR^2$.
Q43. Consider a spherical ball of mass \( m \) and radius \( r \). A thin rod of mass \( m \) and length \( l \) has one of its ends attached to the centre of the sphere and the other end touches the axis of rotation as shown in the figure.

(a) The M.I. of the rod alone about the axis of rotation is \( \frac{ml^2}{3} \)

(b) The M.I. of the sphere alone about the axis of rotation is \( \frac{2}{5}mr^2 + ml^2 \)

(c) The M.I. of the system about the axis of rotation is \( \frac{ml^2}{3} + \left( \frac{2}{5}mr^2 + ml^2 \right) \)

(d) The M.I. of the system about the axis of rotation is \( \frac{ml^2}{5} + \left( \frac{2}{7}mr^2 + 2ml^2 \right) \)

Q44. Consider a solid sphere and hollow sphere of same mass \( M \) and same radius \( R \). Their centres are at a distance \( d \) from each other.

Line \( AB \) bisects \( OO' \) and line \( CD \) is tangent to both sphere then

(a) The moment of inertia of the system about the axis \( AB \) is \( \frac{5}{7}MR^2 + Md^2 \)

(b) The moment of inertia of the system about the axis \( CD \) is \( \frac{8}{16}MR^2 + 3MR^2 \)

(c) The moment of inertia of the system about the axis \( AB \) is \( \frac{16}{15}MR^2 + \frac{Md^2}{2} \)

(d) The moment of inertia of the system about the axis \( CD \) is \( \frac{16}{15}MR^2 + 2MR^2 \)
Q45. Pick out the correct alternative(s)

(a) The radius of gyration of a thin disc about any diameter is \( \frac{R}{2} \), where \( R \) is the radius of the disc.

(b) The radius of gyration of a circular disc about a tangent in its plane is \( \frac{\sqrt{3}}{2} R \), where \( R \) is the radius of the disc.

(c) The radius of gyration of a thin rod about an axis through its one end and perpendicular to the rod is \( \frac{L}{\sqrt{3}} \), where \( L \) is the length of the rod.

(d) The radius of gyration of a rectangular lamina of sides \( l \) and \( b \) about an axis through its centre and perpendicular to its plane is \( \frac{1}{2} \sqrt{\frac{l^2 + b^2}{3}} \).

Q46. A solid sphere of mass \( M \) and radius \( R \) is pulled horizontally on a rough surface. Choose the incorrect alternative(s).

(a) The magnitude of frictional force is \( F \)

(b) The frictional force on the sphere acts forward

(c) The acceleration of the centre of mass is \( \frac{2F}{3M} \)

(d) The acceleration of centre of mass is \( \frac{F}{M} \)

Q47. Consider the uniform solid sphere, sphere (i) has radius \( r \) and mass \( m \), sphere (ii) has radius \( r \) and mass \( 3m \), sphere (iii) has radius \( 3r \) and mass \( m \). All the spheres are placed at the same point on the same inclined plane where they will roll without slipping to reach the bottom. If allowed to roll down the incline, then at the bottom of the incline

(a) sphere (i) will have the largest speed

(b) sphere (ii) will have the largest speed

(c) sphere (ii) will have the largest kinetic energy

(d) all the spheres will have equal speed
Q48. A rigid body is in pure rotation that is undergoing fixed axis rotation. Then which of the following statement(s) are true?

(a) You can find two points in the body in a plane perpendicular to the axis of rotation having the same velocity.

(b) You can find two points in the body in a plane perpendicular to the axis of rotation having the same acceleration.

(c) Speed of all particles lying on the curved surface of a cylinder whose axis coincides with the axis of rotation is the same.

(d) Angular speed of the body is the same as seen from any point in the body.

Q49. The disc of radius \( r \) is confined to roll without slipping at \( A \) and \( B \). If the plates have velocities as shown, then

(a) Angular velocity \( \omega_b \) of the disc is \( \frac{2v}{r} \).

(b) Linear velocity \( v_0 \) of the disc is \( v_0 = v \).

(c) Angular velocity \( \omega_b \) of the disc is \( \frac{3v}{2r} \).

(d) Linear velocity \( v_0 \) of the disc is \( v_0 = 2v \).

Q50. A disc is given an initial angular velocity \( \omega_0 \) and placed gently on a rough horizontal surface as shown in the figure. The quantities which will depend on the coefficient of friction is/are

(a) The time until rolling begins.

(b) The displacement of the disc until rolling begins.

(c) The velocity when rolling begins.

(d) The work done by force of friction.
Q51. A uniform bar of length $6a$ and mass $8m$ lies on a smooth horizontal table. Two point masses $m$ and $2m$ moving in the same horizontal plane with speed $2v$ and $v$ respectively, strike the bar as shown and stick to the bar after collision. Denoting angular velocity, total energy and centre of mass velocity by $\omega, E$ and $v_c$ respectively, we have

(a) $v_0 = 0$  
(b) $\omega = \frac{3v}{5a}$  
(c) $\omega = \frac{v}{5a}$  
(d) $E = \frac{3mv^2}{5}$

Q52. The torque $\vec{\tau}$ on a body about a given by is found to be equal to $\vec{A} \times \vec{L}$ where $\vec{A}$ is a constant vector and $\vec{L}$ is the angular momentum of the body about the given point. From this it follows that

(a) $\frac{d\vec{L}}{dt}$ is perpendicular to $\vec{L}$ at all instant  
(b) The component of $\vec{L}$ in the direction of $\vec{A}$ does not change with time  
(c) The magnitude of $\vec{L}$ does not change with time  
(d) $\vec{L}$ does not change with time

Q53. A sphere is rolling without slipping on a fixed horizontal plane surface. In the figure, $A$ is the point of contact, $B$ is the centre of the sphere and $C$ is its topmost point. Then,

(a) $\vec{v}_C - \vec{v}_A = 2(\vec{v}_B - \vec{v}_C)$  
(b) $\vec{v}_C - \vec{v}_B = \vec{v}_B - \vec{v}_A$  
(c) $|\vec{v}_C - \vec{v}_A| = 2|\vec{v}_B - \vec{v}_C|$  
(d) $|\vec{v}_C - \vec{v}_A| = 4|\vec{v}_B|$
Q54. A particle moves in a circular path with decreasing speed. Choose the correct alternative(s)
   (a) Angular momentum remains constant
   (b) Acceleration $\vec{a}$ is towards the centre
   (c) Particle moves in a spiral path with decreasing radius
   (d) The direction of angular momentum remains constant

Q55. For a rigid body rotating about a fixed axis
   (a) The net torque acting on the body is always along the axis of rotation
   (b) The angular velocity of all the particles have the same value
   (c) The linear acceleration of all the particles have the same value
   (d) The angular momentum and angular velocities always lie in the same direction
MCQ (Multiple Choice Questions)

Ans. 1: (c)
Solution: Mass of the ring \( M = \rho L \)

If \( R \) is the radius of the ring then \( L = 2\pi R \Rightarrow R = \frac{L}{2\pi} \)

M.I of the disc about an axis passing through \( O \) and parallel to \( XX' \) is
\[
I_0 = \frac{MR^2}{4}
\]

hence using parallel axis theorem, M.I about \( XX' \) is
\[
I_{xx'} = I_0 + MR^2 = \frac{MR^2}{4} + MR^2 = \frac{5MR^2}{4} = \frac{5\rho L^2}{4\pi} \quad \text{or} \quad I_{xx'} = \frac{5\rho L^3}{16\pi^2}
\]

Ans. 2: (a)
Solution: The moment of inertia of the complete disc is \( I = \frac{1}{2} (4M) R^2 = 2MR^2 \).

From symmetry each quarter of the disc has the same moment of inertia about the axis of rotation, hence required moment of inertia is
\[
\frac{2MR^2}{4} = \frac{1}{2} MR^2
\]

Ans. 3: (a)
Solution: Total mass of the disc = 9 \( M \)

Radius of disc = \( R \)

Moment of inertia of the complete disc about a perpendicular axis through \( O \) is
\[
I_1 = \frac{1}{2} \times 9M \times R^2 = \frac{9}{2} MR^2
\]

mass of the disc removed = \( \frac{9M}{\pi R^2} \times \left( \frac{R}{3} \right)^2 = M \)

by the parallel axes theorem, the moment of inertia of small disc about the axis through \( O \),
\[ I_2 = \frac{1}{2} M \left( \frac{R}{3} \right)^2 + M \left( \frac{2R}{3} \right)^2 = \frac{1}{2} MR^2 \]

Moment of inertia of the remaining disc about a perpendicular axis through \( O \),
\[ I = I_1 - I_2 = \frac{9}{2} MR^2 - \frac{1}{2} MR^2 = 4MR^2 \]

Ans. 4: (d)

Solution: The moment of inertia of the system about any side (say \( CD \))
\[
I = M.I \text{ of } A \text{ about } CD + \text{ moment of inertia of } B \text{ about } CD + M.I \text{ of } C \text{ about } CD + M.I \text{ of } D \text{ about } CD
\]
\[
= \left( \frac{2}{5} Ma^2 + Mb^2 \right) + \left( \frac{2}{5} Ma^2 + Mb^2 \right) + \frac{2}{5} Ma^2 + \frac{2}{5} Ma^2 = \frac{2}{5} M (4a^2 + 5b^2)
\]

Ans. 5: (b)

Solution: Let \( k_1 \) and \( k_2 \) be the radii of gyration of the ring and the disc respectively. Then
\[
\text{M.I of ring} = MR^2 = Mk_1^2 \text{ or } k_1 = R
\]
\[
\text{M.I of disc} = \frac{1}{2} MR^2 = Mk_2^2 \text{ or } k_2 = \frac{R}{\sqrt{2}} \quad \therefore \frac{k_1}{k_2} = \sqrt{2} : 1
\]

Ans. 6: (d)

Solution: Using parallel axis theorem
\[ I' = I + Mx^2 = \frac{ML^2}{12} + Mx^2 \]

Ans. 7: (d)

Solution: The M.I of the cross about a line perpendicular to the plane of the figure through the centre of cross is
\[ \frac{Ml^2}{12} \times 2 = \frac{Ml^2}{6} \]

The moment of inertia of the cross about the two bisectors are equal by symmetry and according to the theorem of perpendicular axes, the M.I of the cross about the bisector \( AB \) is
\[ \frac{Ml^2}{12} \]
Ans. 8: (c)
Solution: Using parallel axis theorem the M.I of the system about the tangent

\[
= 2 \left[ \frac{2}{5}mr^2 + mr^2 \right] = \frac{14mr^2}{5}
\]

Ans. 9: (c)
Solution: Given that \( I_Q = 4I_P \). The M.I. of a ring about its axis is \( \text{mass} \times (\text{radius})^2 \), let \( \lambda \) be the linear mass density of the wire, then

\[
\lambda \times 2\pi (nr) \times n^2 r^2 = 4 \left( \lambda \times 2\pi r \times r^2 \right) \Rightarrow n^2 = 4 \Rightarrow n = 4^\frac{1}{2}
\]

Ans. 10: (c)
Solution: Let \( M \) be the mass of each disc. Let \( R_A \) and \( R_B \) be the radii of the discs \( A \) and \( B \), respectively. Let \( t \) be the thickness of each disc. Then \( M = \pi R_A^2 \delta t_A = \pi R_B^2 \delta t_B \) as \( \delta t_A > \delta t_B \) so \( R_A^2 < R_B^2 \)

Now \( \frac{I_A}{I_B} = \frac{2}{1} \frac{MR_A^2}{MR_B^2} < 1 \) \( \therefore I_A < I_B \)

Ans. 11: (d)
Solution: Torque \( \vec{\tau} = \vec{r} \times \vec{F} \). For central force \( \vec{r} \) and \( \vec{F} \) lies along the same line. Hence \( \vec{\tau} = 0 \). This implies that angular momentum is constant.

Ans. 12: (b)
Solution: Let \( M \) be the mass and \( r \) be the radius. Then moment of inertia \( \frac{2}{5} Mr^2 \)

Hence \( L = \left( \frac{2}{5} Mr^2 \right) \omega = \frac{2Mr^2 \omega}{5} \)

But since no net torque acts on the sphere angular momentum remains conserved. Hence increasing \( r \), decreases \( \omega \). Increasing \( r \) also increases moment of inertia. Due to changing \( I \) and \( \omega \) rotational kinetic energy also change
Ans. 13: (d)
Solution: Since no external torque acts on the system (Insect + disc), the angular momentum must remain constant. But since $L = I\omega$ and as the insect moves along the diameter $I$ increases, hence $\omega$ must decrease, from centre to rim. From rim to centre $I$ decreases, hence $\omega$ increases.

Ans. 14: (a)
Solution: The net force on the system is zero, hence there will be no translation. But since the net torque is non-zero, there will be rotation.

Ans. 15 (b)
Solution: From the relation
\[
\frac{d\tilde{L}}{dt} = \vec{\tau}_{\text{ext}},
\]
when $\tilde{L}$ is the total angular momentum and $\vec{\tau}$, the net external torque, we see that, when $\vec{\tau}_{\text{ext}} = 0$, $\tilde{L}$ is constant.

Ans. 16: (b)
Solution: The direction of angular velocity vector is always along the axis of rotation for a rigid body rotating about a fixed axis.

Ans. 17: (b)
Solution: About the initial position, the angular momentum of the particle is in the clockwise sense. Also the torque due to its weight $mg$ is in the clockwise sense. Hence the angular momentum increases.

Ans. 18 (a)
Solution: The net velocity is the sum of velocity due to translation and velocity due to rotation.

For rolling we also have $v_{cm} = \omega R$. Hence $v_{top} = v_{cm} + v_{cm} = 2v_{cm}$ and $v_{bottom} = v_{cm} - v_{cm} = 0$.
Ans. 19: (a)

Solution: The acceleration of a rolling body down an incline is given by

\[ a = \frac{g \sin \theta}{1 + \frac{I}{MR^2}} \]

For hollow cylinder \( I/MR^2 = 1 \)

For solid cylinder \( I/MR^2 = \frac{1}{2} \). Hence \( a_{\text{hollow}} < a_{\text{solid}} \)

Since acceleration of solid cylinder is greater if will reach earlier

Ans. 20 (d)

Solution: The torque acting on a particle is \( \vec{\tau} = \vec{r} \times \vec{F} \) and not \( \vec{F} \times \vec{r} \)

The relation \( \vec{\tau} = \frac{d\vec{L}}{dt} \) says zero net torque means constant angular momentum. It also says changing angular momentum necessarily gives rise to torque. Neither zero net force implies zero net torque nor the zero net torque implies zero net force

Ans. 21: (a)

Solution: From the conservation of mechanical energy

\[ Mgl \sin \theta = \frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2 \Rightarrow Mgl \sin \theta = \frac{1}{2} Mv^2 + \frac{1}{2} \left( \frac{MR^2}{2} \right) \frac{v^2}{R^2} \] [For rolling \( v = \omega R \)]

\[ gl \sin \theta = \frac{v^2}{2} + \frac{v^2}{4} \]

\[ \Rightarrow \frac{3v^2}{4} = gl \sin \theta \Rightarrow v = \sqrt{\frac{4gl \sin \theta}{3}} \]

Ans. 22: (d)

Solution: Due to forward slipping \( (v_0 > \omega_0 r) \) the frictional force is backward

The translational of velocity at time \( t \)

\[ v(t) = v_0 - \frac{f}{M} t \] (i)
The angular velocity at time $t$

$$\omega(t) = \frac{v_0}{2r} + \alpha t = \frac{v_0}{2r} + \frac{fr}{2Mr^2} t \Rightarrow \omega(t) = \frac{v_0}{2r} + \frac{5f}{2Mr} t$$

Rolling will start when $v(t) = r\omega(t)$

$$\Rightarrow v(t) = \frac{v_0}{2} + \frac{5f}{2M} t$$ \hspace{1cm} (ii)

Eliminating $t$ from (i) and (ii)

$$\frac{5}{2} v(t) + v(t) = \frac{5}{2} v_0 + \frac{v_0}{2} \text{ or } v(t) = \frac{6v_0}{7}$$

Ans.23: (c)

Solution: Net velocity of a particle = translational velocity + rotational velocity

Hence

$$v = \sqrt{v_0^2 + v^2} = \sqrt{2v_0}$$

Ans.24: (d)

Solution: The forces acting on the seesaw are shown in the figure.

Taking torques about the fulcrum

$$150 \times x = 100 \times 2.5 \text{ which gives } x = 1.7 \text{ m}$$

Ans. 25: (d)

Solution: The velocity of the point of contact of rolling body is zero. Since $v$ and $\omega$ are constants, the only acceleration of the bottommost point is centripetal acceleration $\frac{v^2}{R}$
NAT (Numerical Answer type)

Ans. 26: 140
Solution: The M.I. of the system about the axis = sum of moment of inertia of individual rings.
\[ I = \frac{3}{2}mr^2 + \frac{3}{2}mr^2 + \frac{mr^2}{2} = \frac{7}{2}mr^2 = \frac{7}{2} \times 10 \times 4 = 140 \text{ kg} \cdot \text{m}^2. \]

Ans. 27: 135
Solution: The moment of inertia of the cylinder about its axis \[ \frac{MR^2}{2} \]
Using parallel axis theorem
\[ I = I_{xy} + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2} MR^2 \]
\[ = \frac{3}{2} \times 10 \times 9 = 135 \text{ kg} \cdot \text{m}^2 \]

Ans. 28: 2000
Solution: The moment of inertia of the disc about axis \[ AB \] is
\[ I_{AB} = I_{xy} + MR^2 = \frac{MR^2}{4} + MR^2 = \frac{5}{4} MR^2 \]
The angular speed \( \omega \) of the disc = \[ \frac{v}{R} \]
Hence \[ K = \frac{1}{2} \times \frac{5}{4} MR^2 \times \frac{v^2}{R^2} = \frac{1}{2} \times \frac{5}{4} \times 8 \times 400 = 2000 \text{ J} \]

Ans. 29: 50
Solution: The moment of inertia of the system about the altitude passing through \( m_1 \) is
\[ I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 \]
\[ = (4 \text{ kg})(0)^2 + (3 \text{ kg})(2.5 \text{ m})^2 + (5 \text{ kg})(2.5 \text{ m})^2 \]
\[ = (2.5)^2 \times 8 = 50 \text{ kg} \cdot \text{m}^2 \]
Ans. 30: 64

Solution: The moment of inertia of the plate about an axis perpendicular to the plate and passing through centre $O$ is

$$I = \frac{m}{12}(a^2 + a^2) = \frac{ma^2}{6}$$

By parallel axis theorem, the moment of inertia of the plate about an axis through one of its corners,

$$I_A = I_o + m\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{ma^2}{6} + \frac{ma^2}{2} = \frac{2}{3}ma^2$$

$$= \frac{2}{3} \times 6 \times 6 = 64 \text{ kg} \cdot \text{m}^2$$

Ans. 31: $2\sqrt{26}$

Solution: The angular momentum of the disc

$\vec{L} = \vec{L}_{cm} + M(\vec{R} \times \vec{V})$ when $\vec{L}_{cm}$ is the angular momentum of the disc with respect to centre of mass, $\vec{R}$ the position of centre of mass and $\vec{V}$ the velocity of centre of mass

$\vec{L}_{cm} = I_{cm}\omega$

or $\vec{L}_{cm} = \frac{1}{2}(2)(1)^2(-10\hat{k}) = -10\hat{k}$

$M(\vec{R} \times \vec{V}) = 2[\hat{x}(\vec{x}) + \hat{y}(\vec{y}) \times (10 \times 1\hat{i})] = -2\hat{y} = -2 \times 1\hat{k} = -2\hat{k}$

Hence $L = \sqrt{(-10)^2 + (-2)^2} = \sqrt{104} = 2\sqrt{26}$ units

Ans. 32: 5.98

Solution: The acceleration of a rolling body down an incline is given by the formula

$$a = \frac{g \sin \theta}{1 + \frac{I}{M R^2}}$$
Ans. 33: 100

Solution: From the conservation of mechanical energy
\[
\frac{1}{2} M v^2 + \frac{1}{2} I \omega^2 = 0 + 0 + Mgh
\]
or
\[
\frac{1}{2} M v^2 + \frac{1}{2} \frac{2}{5} MR^2 \frac{v^2}{R^2} = Mgh
\]
\[
\frac{v^2}{2} + \frac{v^2}{5} = gh \Rightarrow 7v^2 = 10gh
\]
\[
\therefore v = \sqrt{\frac{10gh}{7}}
\]
or
\[
v = \sqrt{\frac{10 \times 10 \times 700}{7}} = 100 \text{ m/s}
\]

Ans. 34: 200 N

Solution: Moments of normal reaction and frictional force about the point of tipping O is zero. At the time of tipping N passes through O. To tip about the edge, moment of F should be greater than the moment of mg

or \[ F \left( \frac{3a}{4} \right) > mg \left( \frac{a}{2} \right) \text{ or } F > \frac{2}{3} mg \]

Hence \[ F > \frac{2}{3} \times 30 \times 10 \Rightarrow F > 200 N \]

Ans. 35: \(-3\)

Solution: \[ mvr = k \text{ (a constant)} \Rightarrow v = \frac{k}{mr} \]

\[
T = \frac{mv^2}{r} = \frac{m}{r} \left( \frac{k}{mr} \right)^2 = \frac{k^2}{m} \cdot \frac{1}{r^3}
\]
\[ T = Ar^{-3} \quad \text{(where } A = \frac{k^2}{m} \text{)} \]

Hence \( n = -3 \)

Ans. 36: \( \frac{40}{3} \)

Solution: Taking torque about point \( B \)

\[ W(d-x) = N_A d \]

\[ \Rightarrow N_A = \left( \frac{d-x}{d} \right) w = \frac{30-10}{30} \times 20 \Rightarrow N_A = \frac{40}{3} N \]

Ans. 37: 8

Solution: Since no net external torque acts on the system, hence, the angular momentum about the axis of rotation remains conserved. Initially the two discs rotate with different angular speeds but due to friction they acquire common angular speed. From \( L = I\omega \) we have \( I_i\omega_i = I_f\omega_f \)

\[ \Rightarrow \frac{1}{2} MR^2 (10) = \left[ \frac{1}{2} MR^2 + \frac{1}{2} \frac{M}{4} R^2 \right] \omega_f \]

\[ 5MR^2 = \frac{5MR^2}{8} \omega_f \]

or \( \omega_f = 8 \text{ rad/s} \)

Ans. 38: 6

Solution: Angular momentum of the system about an axis perpendicular to the plane of paper and passing through \( O \) will remains conserved.

\[ L_i = L_f \]

\[ \therefore mvl = I\omega = \left( mL^2 + \frac{ML^2}{3} \right) \omega \]

\[ \therefore \omega = \frac{3mv}{L(3m + M)} = \frac{3 \times 1 \times 10}{1(3 + 2)} \]

\[ \omega = 6 \text{ rad/s} \]
Ans. 39: 4
Solution: For the block \( mg - T = ma \)  
\[
(i)
\]
For the pulley
\[
TR = I \alpha, \quad \alpha \text{ being the angular acceleration}
\]
\[
\Rightarrow TR = \frac{MR^2}{2} \cdot a \Rightarrow T = \frac{Ma}{2} \quad (ii)
\]
From (i) and (ii) \( 2g - \frac{Ma}{2} = ma \)
\[
\Rightarrow a \left[ m + \frac{M}{2} \right] = mg \Rightarrow a \left( \frac{2m + M}{2} \right) = mg
\]
\[
a = \frac{2mg}{2m + M} = \frac{2 \times 2 \times 10}{4 + 6} = 4 \text{ m/s}^2
\]
Ans. 40: 1
Solution: The tension in the rope is the only net force on the block and it does not exert any torque about the axis of rotation. Hence, the angular momentum of the block about the axis should remain conserved.

Hence \( m_1v_1r_1 = m_2v_2r_2 \)  
\[
(i)
\]
When \( r_2 \) and \( v_2 \) are the radius and speed when the string breaks. Let \( T_2 \) be the tension when string breaks.

Hence \( T_2 = \frac{mv_2^2}{r_2} \)  
\[
(ii)
\]
From (i) and (ii) \( r_2 = \left( \frac{mv_1^2r_1^2}{T_2} \right)^{\frac{1}{3}} \)
\[
r_2 = \left( \frac{150 \times 16 \times 0.25}{600} \right)^{\frac{1}{3}} \Rightarrow r_2 = 1 \text{ m}
MSQ (Multiple Select Questions)

Ans. 41: (a), (b) and (c)
Solution: From symmetry consideration and perpendicular axes theorem

\[ I_1 = I_2 = I_3 = I_4 = \frac{ML^2}{12} \]

Ans. 42: (a), (b) and (c)
Solution: The moment of inertia of the disc about point \( O \)

Moment of inertia about \( AB \) is

\[ I_{AB} = I_{AB}' + M \left( \frac{R}{2} \right)^2 = \frac{MR^2}{4} + \frac{MR^2}{4} = \frac{MR^2}{2} \]

\[ I_{CD} = I_{CD}' + MR^2 = \frac{MR^2}{4} + MR^2 = \frac{5}{4}MR^2 \]

Ans. 43: (a), (b) and (c)
Solution: From parallel axis theorem, M.I. of the rod about \( XY \) is

\[ I_{rod} = I_{AB} + m \left( \frac{l}{2} \right)^2 = \frac{ml^2}{12} + \frac{ml^2}{4} \]

\[ \Rightarrow I_{rod} = \frac{ml^2}{3} \]

The moment of inertia of the sphere (using parallel axis theorem) about \( XY \) is

\[ I_{sphere} = I_{CD} + ml^2 \]

\[ \Rightarrow I_{sphere} = \frac{2}{5}mr^2 + ml^2 \]

The moment of inertia of the system about \( XY \) is

\[ I_{system} = \frac{ml^2}{3} + \frac{2}{5}mr^2 + ml^2 = \frac{ml^2}{3} + \left( \frac{2}{5}mr^2 + ml^2 \right) \]
Ans. 44: (c) and (d)

Solution: Moment of inertia of the system about $AB = \text{Moment of inertia of the hollow sphere about } AB$ + Moment of inertia of solid sphere about $AB$

$$I_{AB} = \left[ \frac{2}{3}MR^2 + M\left(\frac{d}{2}\right)^2 \right] + \left[ \frac{2}{5}MR^2 + M\left(\frac{d}{2}\right)^2 \right]$$

$$= \frac{16}{15}MR^2 + \frac{Md^2}{2}$$

Moment of inertia of the system about $CD = \text{Moment of inertia of the hollow sphere about } CD + \text{Moment of inertia of the solid sphere about } CD$.

$$\Rightarrow I_{CD} = \left[ \frac{2}{3}MR^2 + MR^2 \right] + \left[ \frac{2}{5}MR^2 + MR^2 \right] = \frac{16}{15}MR^2 + 2MR^2$$

Ans. 45: (a), (b), (c) and (d)

Solution: For the disc about its diameter $Mk^2 = \frac{MR^2}{4} \Rightarrow k = \frac{R}{2}$

For the disc about the tangent in its plane $Mk^2 = \frac{MR^2}{4} + MR^2 = \frac{5}{4}MR^2 \Rightarrow k = \frac{\sqrt{5}}{2}R$

For the thin rod about an axis through one end $MK^2 = \frac{ML^2}{3} \Rightarrow K = \frac{L}{\sqrt{3}}$

For a rectangular lamina about an axis through the centre and perpendicular to the rod $Mk^2 = \frac{M}{12}(l^2 + b^2) \Rightarrow k = \frac{1}{2} \sqrt{\frac{l^2 + b^2}{3}}$

Ans. 46: (a), (b), (c) and (d)

Solution: The force $F$ tries to pull the point of contact in the forward direction, hence the frictional force is backward.

Translation equation

$$F - f = Ma \quad (i)$$

Rotational equation about the centre of mass
\[ fR = Iα = \frac{2MR^2}{5} \cdot \frac{a}{R} \]

\[ \Rightarrow f = \frac{2Ma}{5} \quad \text{(ii)} \]

From (i) and (ii) \[ F = \frac{7Ma}{5} \] or \[ a = \frac{5F}{7M} \]

and \[ f = \frac{2M}{5} \cdot \frac{5F}{7M} = \frac{2F}{7} \]

Ans. 47: (c) and (d)

Solution: Consider an arbitrary sphere of mass \( M \) and radius \( R \) rolling down an incline from height \( h \).

From conservation of mechanical energy

\[ Mgh = \frac{1}{2} Mv^2 + \frac{1}{2} Iω^2 \]

\[ \Rightarrow Mgh = \frac{1}{2} Mv^2 + \frac{1}{2} \cdot \frac{2MR^2}{5} \cdot \frac{v^2}{R^2} \]

\[ \Rightarrow gh = \frac{7v^2}{10} \quad \text{or} \quad v = \frac{\sqrt{\frac{10gh}{7}}}{v} \]

This is independent of mass and radius thus all the spheres reach with the same speed.

The total kinetic energy of the spheres at the bottom is equal to the loss in initial potential energy. Hence

\[ K = Mgh \Rightarrow K \propto M \]

since sphere (ii) has the largest mass if will have the largest kinetic energy

Ans. 48: (c) and (d)

Solution: All points in the body, in plane perpendicular to axis of rotation, revolve in concentric circles.

All points lying on the circle of same radius have same speed (and also same magnitude of acceleration) but different directions of velocity.

Angular speed of body of any instant with respect to any point in the body is the same by definition.
Ans. 49: (a) and (b)

Solution: In case of rolling the point of contact and the surface on which the object is rolling should have same velocity.

Hence \( v_0 - r\omega_0 = 3v \) \( \text{(i)} \)

and \( v_0 + r\omega_0 = -v \) \( \text{(ii)} \)

From (i) and (ii) \( v_0 = v \) and \( \omega_0 = \frac{2v}{r} \)

Ans. 50: (a) and (b)

Solution:

\[
\frac{f}{M} t \quad \text{and} \quad \frac{fR}{MR^2} t
\]

\[
v(t) = \frac{\mu_0 Mg}{M} t = \mu_0 gt \quad \text{or} \quad \omega(t) = \omega_0 - \frac{2\mu_0 MgR}{MR^2} t = \omega_0 - \frac{2\mu_0 g}{R} t
\]

when rolling being

\[
v(t) = R\omega(t) \Rightarrow \mu_0 gt = \omega_0 R - 2\mu_0 gt \Rightarrow t = \frac{\omega_0 R}{3\mu_0 g}
\]

velocity at the time of rolling

\[
v(t) = \mu_0 g \cdot \frac{\omega_0 R}{3\mu_0 g} = \frac{\omega_0 R}{3}
\]

displacement = \( \frac{1}{2} \mu_0 g \left( \frac{\omega_0 R}{3\mu_0 g} \right)^2 = \frac{\omega_0^2 R^2}{18\mu_0 g} \)

work done by force of friction = \( (-\mu_0 Mg) \times \frac{\omega_0^2 R^2}{18\mu_0 g} = -\frac{\omega_0^2 R^2 M}{18} \)

Ans. 51: (a), (c) and (d)

Solution: Since initial linear momentum of the system is zero, and no external force acts on the system,

hence from \( \vec{P} = M\vec{v}_c \), \( v_c = 0 \)

Since no external torque acts on the system \( L_i = L_f \)

\( \Rightarrow (2mv)a + (2mv)2a = 1\omega \)
\[ 6mva = I\omega \]  \hspace{2cm} \text{(i)}

\[ I = \frac{(8m)(6a)^2}{12} + m(2a)^2 + 2m(a^2) = 30ma^2 \]

hence from equation (i) \[ \omega = \frac{v}{5a} \]

Further, \[ E = \frac{1}{2} I\omega^2 = \frac{1}{2} \cdot 30ma^2 \cdot \frac{v^2}{25a^2} = \frac{3mv^2}{5} \]

Ans. 52: (a), (b) and (c)

Solution: \[ \vec{a} = \vec{A} \times \vec{L} \text{ i.e. } \frac{d\vec{L}}{dt} = \vec{A} \times \vec{L} \]

This implies that \[ \frac{d\vec{L}}{dt} \] is perpendicular to both \[ \vec{A} \times \vec{L} \] of all instants.

We have \[ \vec{L} \cdot \vec{L} = L^2 \] \[ \Rightarrow \vec{L} \cdot \frac{d\vec{L}}{dt} + \frac{d\vec{L}}{dt} \cdot \vec{L} = 2\vec{L} \frac{d\vec{L}}{dt} \]

then \[ \frac{d\vec{L}}{dt} = 0 \]

or magnitude of \[ \vec{L} \] does not change with time. Similarly we can show that \[ \vec{L} \perp \vec{A} \]. Thus the component of \[ \vec{L} \] along \[ \vec{A} \] is always constant.

Ans. 53: (b) and (c)

Solution: \[ \vec{v}_A = 0, \quad \vec{v}_b = \vec{v}, \quad \vec{v}_c = 2\vec{v} \]

Ans. 54: (d)

Solution: The magnitude of angular momentum of any instant is \[ L = mvr \]

Since \[ r \] is the same at all time and \[ v \] is decreasing, \[ L \] is decreasing. The acceleration \[ \vec{a} \] is towards the centre only when the particle moves in the circle with constant speed.

From the figure, we see that \[ \vec{r} \times \vec{P} \] is always out of paper, hence direction of \[ \vec{L} \] remains constant.
Ans. 55: (a) and (b)

Solution: For a fixed axis rotation, the net torque is always along the axis of rotation, otherwise the axis itself would not remain fixed.

All the particles of the rigid body moves equal angles in equal time. Hence angular velocity of all particles in the same

\[ a \]

\[ \alpha R \]

\[ \omega^2 R \]

The linear acceleration \( a \) depends on the distance of the particle from the axis of rotation.

The angular momentum vector lies in the direction of angular velocity only if the axis rotation is one of the symmetry axes of the body.
5. Fluid Mechanics

Fluids

Matter is broadly divided into three categories, solid, liquid and gas. The intermolecular forces are strong in solids, so that the shape and size of solids do not easily change. This force is comparatively less in liquids and so the shape is easily changed. Although the shape of a liquid can be easily changed, the volume of a given mass of a liquid is not so easy to change. It needs quite a good effort to change the density of liquids. In gases, the intermolecular forces are very small and it is simple to change both the shape and the density of a gas.

We shall assume that the liquids we deal with are incompressible and non viscous. The first condition means that the density of the liquid is independent of the variations in pressure and always remains constant. The second condition means that parts of the liquid in contact do not exert any tangential force on each other. The force by one part of the liquid on the other part is perpendicular to the surface of contact. Thus, there is no friction between the adjacent layers of a liquid.

5.1 Pressure in a Fluid

Consider a point $A$ in the fluid (figure 1). Imagine a small area $\Delta S$ containing the point $A$. The fluid on one side of the area presses the fluid on the other side and vice versa. Let the common magnitude of the forces be $F$. Then the pressure of the fluid at the point $A$ as

$$P = \lim_{\Delta S \to 0} \frac{F}{\Delta S} \quad \ldots(1)$$

For a homogeneous and nonviscous fluid, this quantity does not depend on the orientation of $\Delta S$ and hence we talk of pressure at a point. For such a fluid, pressure is a scalar quantity having only magnitude.

Unit of Pressure

The SI unit of pressure is $N/m^2$ called Pascal $(Pa)$.

![Figure 1](image-url)
5.1.1 Variation of Pressure with Height

Let us consider two points \( A \) and \( B \) (figure 2) separated by a small vertical height \( dz \). Imagine a horizontal area \( \Delta S_1 \) containing \( A \) and an identical horizontal area \( \Delta S_2 \) containing \( B \). The area \( \Delta S_1 = \Delta S_2 = \Delta S \). Consider the fluid enclosed between the two surfaces \( \Delta S_1, \Delta S_2 \) and the vertical boundary joining them. The vertical forces acting on this fluid are

(a) \( F_1 \), vertically upward by the fluid below it
(b) \( F_2 \), vertically downward by the fluid above it and
(c) weight \( W \), vertically downward

Let the pressure at the surface \( A \) be \( P \) and the pressure at \( B \) be \( P + dP \). Then

\[
F_1 = P \Delta S \quad \text{and} \quad F_2 (P + dP) \Delta S.
\]

The volume of the fluid considered is \( \Delta S (dz) \). If the density of the fluid at \( A \) is \( \rho \), the mass of the fluid considered is \( \rho (\Delta S) (dz) \) and hence its weight \( W \) is \( W = \rho (\Delta S) (dz) g \).

For vertical equilibrium,

\[
F_1 = F_2 + W
\]

or

\[
P \Delta S = (P + dP) \Delta S + \rho g (dz) \Delta S
\]

or

\[
dP = -\rho g (dz).
\] ... (2)

As we move up through a height \( dz \) the pressure decreases by \( \rho g dz \) where \( \rho \) is the density of the fluid at that point.

Now consider two points at \( z = 0 \) and \( z = h \). If the pressure at \( z = 0 \) is \( P_1 \) and that at \( z = h \) is \( P_2 \), then from equation (2),

\[
\int_{P_1}^{P_2} dP = \int_0^h -\rho g dz \Rightarrow P_2 - P_1 = -\rho g h
\]

\[
\Rightarrow P_1 = P_2 + \rho g h \quad \text{....(3)}
\]
Next consider two points \(A\) and \(B\) in the same horizontal line inside a fluid. Imagine a small vertical area \(\Delta S_1\) containing the point \(A\) and a similar vertical area \(\Delta S_2\) containing the point \(B\).

The area \(\Delta S_1 = \Delta S_2 = \Delta S\). Consider the contained in the horizontal cylinder bounded and \(\Delta S_1\) and \(\Delta S_2\). If the pressures at \(A\) and \(B\) are \(P_1\) and \(P_2\) respectively, the forces in the direction \(AB\) are

(a) \(P_1\Delta S\) towards right and
(b) \(P_2\Delta S\) towards left.

If the fluid remains in equilibrium, \(P_1\Delta S = P_2\Delta S\) or \(P_1 = P_2\).

Thus, the pressure is same at two in the same horizontal level.

5.1.2 Pascal’s Law

We have seen in the previous section that the pressure difference between two points in a liquid at rest depends only on the difference in vertical height between the points. The difference is in fact \(\rho gz\), where \(\rho\) is the density of the liquid (assumed constant) and \(z\) is the difference in vertical height.

If the pressure in a liquid is changed at a particular point, the change is transmitted to the entire liquid without being diminished in magnitude.

As an example, suppose a flask fitted with a piston is filled with a liquid as shown in figure (4). Let an external force \(F\) be applied on the piston. If the cross-sectional area of the piston is \(A\), the pressure just below the piston is increased by \(\frac{AF}{A}\). By Pascal’s law, the pressure at any point \(B\) will also increase by the same amount \(\frac{AF}{A}\).

This is because the pressure at \(B\) has to be \(\rho gz\) more than the pressure at the piston, where \(z\) is the vertical distance of \(B\) below the piston.

By applying the force we do not appreciably change \(z\) (as the liquid is supposed to be incompressible) and hence the pressure difference remains unchanged. As the pressure at the piston is increased by \(\frac{AF}{A}\), the pressure at \(B\) also increases by the same amount.
5.1.3 Atmospheric Pressure and Barometer

The atmosphere of the earth is spread up to a height of about 200 km. This atmosphere presses the bodies on the surface of the earth. The force exerted by the air on any body is perpendicular to the surface of the body. Consider a small surface $\Delta S$ in contact with air. If the force exerted by the air on this part is $F$, the atmospheric pressure is $P_0 = \lim_{\Delta S \to 0} \frac{F}{\Delta S}$.

Atmospheric pressure at the top of the atmosphere is zero as there is nothing above it to exert the force. The pressure at a distance $z$ below the top will be $\int_0^z \rho g \, dz$. Remember, neither $\rho$ nor $g$ can be treated as constant over large variations in heights. However, the density of air is quite small and so the atmospheric pressure does not vary appreciably over small distances. Thus, we say that the atmospheric pressure at Patna is 76 cm of mercury without specifying whether it is at Gandhi Maidan or at the top of Golghar.

Torricelli devised an ingenious way to measure the atmospheric pressure. The instrument is known as barometer. In this, a glass tube open at one end and having a length of about a meter is filled with mercury. The open end is temporarily closed (by a thumb or otherwise) and the tube is inverted in a cup of mercury. With the open end dipped into the cup, the temporary closure is removed. The mercury column in the tube falls down a little and finally stays there.

Figure (5) shows schematically the situation. The upper part of the tube contains vacuum as the mercury goes down and no air is allowed in.

Thus, the pressure at the upper end $A$ of the mercury column inside the tube is $P_A = 0$. Let us consider a point $C$ on the mercury surface in the cup and another point $B$ in the tube at the same horizontal level. The pressure at $C$ is equal to the
atmospheric pressure. As $B$ and $C$ are in the same horizontal level, the pressures at $B$ and $C$ are equal. Thus, the pressure at $B$ is equal to the atmospheric pressure $P_0$ in the lab.

Suppose the point $B$ is at a depth $H$ below $A$. If $\rho$ be the density of mercury,

$$P_B = P_A + \rho g H \quad \text{or}, \quad P_0 = \rho g H. \quad \ldots \ (4)$$

The height $H$ of the mercury column in the tube above the surface in the cup is measured. Knowing the density of mercury and the acceleration due to gravity, the atmospheric pressure can be calculated using equation (4).

The atmospheric pressure is often given as the length of mercury column in a barometer. Thus, a pressure of 76 cm of mercury means

$$P_0 = \left(13.6 \times 10^3 \text{ kg/m}^3\right) \left(9.8 \text{ m/s}^2\right) \times 0.76 \text{ m}$$

$$= 1.01 \times 10^5 \text{ Pa.}$$

This pressure is written as 1 atm. If the tube is insufficient in length, the mercury column will not fall down and no vacuum will be created. The inner surface of the tube will be in contact with the mercury at the top and will exert a pressure $P_A$ on it.

**Example:** Water is filled in a flask up to a height of 20 cm. The bottom of the flask is circular with radius 10 cm. If the atmospheric pressure is $1.01 \times 10^6 \text{ Pa}$, find the force exerted by the water on the bottom. Take $g = 10 \text{ m/s}^2$ and density of water $= 1000 \text{ kg/m}^3$.

**Solution:** The at the surface of the water is equal to the atmospheric pressure $P_0$. The pressure at the bottom is

$$P = P_0 + h \rho g = 1.01 \times 10^6 \text{ Pa} + \left(0.20 \text{ m}\right) \left(1000 \text{ kg/m}^3\right) \left(10 \text{ m/s}^2\right)$$

$$= 1.01 \times 10^6 \text{ Pa} + 0.02 \times 10^5 \text{ Pa} = 1.03 \times 10^6 \text{ Pa}.$$

The area of the bottom

$$= \pi r^2 = 3.14 \times \left(0.1 \text{ m}\right)^2 = 0.0314 \text{ m}^2$$

The force on the bottom is, therefore, $F = P \pi r^2$

$$= \left(1.03 \times 10^6 \text{ Pa}\right) \times \left(0.0314 \text{ m}^2\right) = 3230 \text{ N}$$. 
5.1.4 Manometer

Manometer is a simple device to measure the pressure in a closed vessel containing a gas. It consists of a \( U \)-tube having some liquid. One end of the tube is open to the atmosphere and the other end is connected to the vessel (figure 6).

The pressure of the gas is equal to the pressure at \( A \) 
\[ \text{pressure at } B = \text{pressure at } C + h \rho g = P_0 + h \rho g \]
where \( P_0 \) is the atmospheric pressure, \( h = BC \) is the difference in levels of the liquid in the two arms and \( \rho \) is the density of the liquid.

The excess pressure \( P - P_0 \) is called the gauge pressure.

5.2 Archimedes’ Principle

When a body is partially or fully dipped into a fluid, the fluid exerts forces on the body. At any small portion of the surface of the body, the force by the fluid is perpendicular to the surface and is to the pressure at that point multiplied by the area (figure 8). The resultant of all these contact forces is called the force of buoyancy or buoyant force.

Archimedes’ principle states that when a body is partially or fully dipped into a fluid at rest, the fluid exerts an upward force of buoyancy equal to the weight of the displaced fluid.

Archimedes’ principle is not an independent principle and may be deduced from Newton’s laws of motion.

Consider the situation shown in figure (7) where a body is shown dipped into a fluid. Suppose the body dipped in the fluid is replaced by the same ‘fluid of equal volume. As the entire fluid now becomes homogeneous, all parts will remain in equilibrium. The part of the fluid substituting the body also remains in equilibrium.
Forces acting on this substituting fluid are
(a) the weight $mg$ of this part of the fluid and
(b) the resultant $B$ of the contact forces by the remaining fluid.

As the substituting fluid is in equilibrium, these two should be equal and opposite. Thus,

$$B = mg \quad \ldots \ (5)$$

and it acts in the vertically upward direction.

**Floatation**

When a solid body is dipped into a fluid, the fluid exerts an upward force of buoyancy on the solid. If the force of buoyancy equals the weight of the solid, the solid will remain in equilibrium. This is called floatation. When the overall density of the solid is smaller than the density of the fluid, the solid floats with a part of it in the fluid. The fraction dipped is such, that the weight of the displaced fluid equals the weight of the solid.

**Example:** A 700 $g$ solid cube having an edge of length 10 $cm$ floats in water. How much volume of the cube is outside the water? Density of water $= 1000 \frac{kg}{m^3}$.

**Solution:** The weight of the cube is balanced by the buoyant force. The buoyant force is equal to the weight of the water displaced. If a volume $V$ of the cube is inside the water, the weight of the displaced water $= \rho V g$, where $\rho$ is the density of water. Thus,

$$V \rho g = (0.7 \ kg \)g$$

or

$$V = \frac{0.7 \ kg}{\rho} = \frac{0.7 \ kg}{1000 \frac{kg}{m^3}} = 7 \times 10^{-4} \ m^3 = 700 \ cm^3$$

The total volume of the cube $= (10 \ cm)^3 = 1000 \ cm^3$

The volume outside the water is

$$100 \ cm^3 - 700 \ cm^3 = 300 \ cm^3.$$
5.2.1 Pressure Difference and Buoyant Force in Accelerating Fluids

Equations (3) and (5) were derived by assuming that the fluid under consideration is in equilibrium in an inertial frame. If this is not the case, the equations must be modified. We shall discuss some special cases of accelerating fluids.

5.2.2 A Liquid Placed in an Elevator

(a) Pressure Difference

Suppose a beaker contains some liquid and it is placed in an elevator which is going up with acceleration \( a_0 \) (figure 8). Let \( A \) and \( B \) be two points in the liquid, \( B \) being at a vertical height \( z \) above \( A \). Construct a small horizontal area \( \Delta S \) around \( A \) and an equal horizontal area around \( B \). Construct a vertical cylinder with the two areas as the faces. Consider the motion of the liquid contained within this cylinder. Let \( P_1 \) be the pressure at \( A \) and \( P_2 \) be the pressure at \( B \).

Forces acting on the liquid contained in the cylinder, in the vertical direction, are

(a) \( P_1 \Delta S \), upward due to the liquid below it
(b) \( P_2 \Delta S \), downward due to the liquid above it and
(c) weight \( mg = (\Delta S) \varepsilon \rho g \) downward, where \( \rho \) is the density of the liquid.

Under the action of these three forces the liquid is accelerating upward with an acceleration \( a_0 \). From Newton’s second law

\[ P_1 \Delta S - P_2 \Delta S - mg = ma_0 \]

or

\[ (P_1 - P_2) \Delta S = m(g + a_0) = (\Delta S) \varepsilon \rho (g + a_0) \]

or

\[ P_1 - P_2 = \rho (g + a_0) \varepsilon. \]  \hspace{1cm} \ldots (6)
(b) Buoyant Force

Now suppose a body is dipped inside a liquid of density $\rho$ placed in an elevator going up with acceleration $a_0$. Let us calculate the force of buoyancy $B$ on this body. As was done earlier, let us suppose that we substitute the body into the liquid by the same liquid of equal volume. The entire liquid becomes a homogenous mass and hence the substituted liquid is at rest with respect to the rest of the liquid. Thus, the substituted liquid is also going up with an acceleration $a$ together with the rest of the liquid.

The forces acting on the substituted liquid are
(a) the buoyant force $B$ and
(b) the weight $mg$ of the substituted liquid. From Newton’s second law,

From Newton’s second law,

$$B - mg = ma_0 \text{ or } B = m(g + a_0) \quad \ldots \ldots (7)$$

5.2.3 Free Surface of a Liquid in Horizontal Acceleration

Consider a liquid placed in a beaker which is accelerating horizontally with acceleration $a_0$ (figure 9). Let $A$ and $B$ be two points in the liquid at a separation $l$ in the same horizontal line along the acceleration $a_0$. We shall first obtain the pressure difference between the points $A$ and $B$.

Construct a small vertical area $\Delta S$ around $A$ and an equal area around $B$. Consider the liquid contained in the horizontal cylinder with the two areas as the flat faces. Let the pressure at $A$ be $P_1$ and the pressure at $B$ be $P_2$. The forces along the line $AB$ are

(a) $P_1\Delta S$ towards right due to the liquid on the left and

(b) $P_2\Delta S$ towards left due to the liquid on the right.

Under the action of these forces, the liquid contained in the cylinder is accelerating towards right. From Newton’s second law,
\[ P_1 \Delta S - P_2 \Delta S = ma_0 \quad \text{or} \quad (P_1 - P_2) \Delta S = (\Delta S)l \rho a_0 \]

or
\[ P_1 - P_2 = l \rho a_0 \quad \text{.... (8)} \]

The two points in the same horizontal line do not have equal pressure if the liquid is accelerated horizontally.

As there is no vertical acceleration, the equation (3) is valid. If the atmospheric pressure is \( P_0 \), the pressure at \( A \) is \( P_1 = P_0 + h_1 \rho g \) and the pressure at \( B \) is \( P_2 = P_0 + h_2 \rho g \), where \( h_1 \) and \( h_2 \) are the depths of \( A \) and \( B \) from the free surface. Substituting in (8),

\[ h_1 \rho g - h_2 \rho g = l \rho a_0 \quad \text{or} \quad \frac{h_1 - h_2}{l} = \frac{a_0}{g} \quad \text{or} \quad \tan \theta = \frac{a_0}{g} \]

where \( \theta \) is the inclination of the free surface with the horizontal.

5.3 Steady and Turbulent Flow

Consider a liquid passing through a glass tube (figure 10). Concentrate on a particular point \( A \) in the tube and look at the particles arriving at \( A \). If the velocity of the liquid is small, all the particles which come to \( A \) will have same speed and will move in same direction. As a particle goes from \( A \) to another point \( B \), its speed and direction may change, but all the particles reaching \( A \) will have the same speed at \( A \) and all the particles reaching \( B \) will have the same speed at \( B \). Also, if one particle passing through \( A \) has gone through \( B \), then all the particles passing through \( A \) go through \( B \). Such a flow of fluid is called a steady flow.

In steady flow the velocity of fluid particles reaching a particular point is the same at all time. Thus each particle follows the same path as taken by a previous particle passing through that point.

If the liquid is pushed in the tube at a rapid rate, the flow may become turbulent. In this case, passing through the same point may be different with and change erratically with time. The motion of water in a high fall or a fast flowing river is, in general, turbulent. Steady flow is also called streamline flow.
5.3.1 Line of Flow: Streamline

The path taken by a particle in flowing fluid is called its line of flow. The tangent at any point on the line of flow gives the direction of motion of that particle at that point. In the case of steady flow, all the particles passing through a given point follow the same path and hence we have a unique line of flow passing through a given point. In this case, the line of flow is also called a streamline.

5.3.2 Irrotational flow of an incompressible and non-viscous fluid

The analysis of the flow of a fluid becomes much simplified if we consider the fluid to be incompressible and non-viscous and that the flow is irrotational. Incompressibility means that the density of the fluid is same at all the points and remains constant as time passes. Viscosity of a fluid is related to the internal friction when a layer of fluid slips over another layer. Mechanical energy is lost against such viscous forces. The assumption of a nonviscous fluid will mean that we are neglecting the effect of such internal friction. Irrotational flow means there is no net angular velocity of fluid particles. When you put some washing powder in a bucket containing water and mix it by rotating your hand in circular path along the wall of the bucket, the water comes into rotational motion.

5.4 Equation of Continuity

We have seen that the fluid going through a tube of flow does not intermix with fluid in other tubes. The total mass of fluid going into the tube through any cross-section should, therefore, be equal to the total mass coming out of the same tube from any other cross-section in the same time. This leads to the equation of continuity.

Let us consider two cross-sections of a tube of flow at the points $A$ and $B$ (figure 11). Let the area of cross-section at $A$ be $A_1$ and that at $B$ be $A_2$. Let the speed of the fluid be $v_1$ at $A$ and $v_2$ at $B$.

![Figure 11](image-url)
How much fluid goes into the tube through the cross-section at $A$ in a time interval $\Delta t$?

Let us construct a cylinder of length $u$ at $A$ as shown in the figure. As the fluid at $A$ has speed $v$ all the fluid included in this cylinder will cross through $A_1$ in the time interval $\Delta t$. Thus, the volume of the fluid going into the tube through the cross-section at $A$ is $A_1 v_1 \Delta t$. Similarly, the volume of the fluid going out of the tube through the cross-section at $B$ is $A_2 v_2 \Delta t$. If the fluid is incompressible, we must have

$$A_1 v_1 \Delta t = A_2 v_2 \Delta t$$

or

$$A_1 v_1 = A_2 v_2 \quad \ldots \quad (9)$$

The product of the area of cross-section and the speed remains the same at all points of a tube of flow. This is called the equation of continuity and expresses the law of conservation of mass in fluid dynamics.

**Example:** Figure below shows a liquid being pushed out of a tube by pressing a piston. The area of cross-section of the piston is $1.0 \text{cm}^2$ and that of the tube at the outlet is $20 \text{mm}^2$. If the piston is pushed at a speed of $2 \text{cm/s}$ what is the speed of the outgoing liquid?

![Diagram of liquid being pushed out of a tube](image)

**Solution:** From the equation

$$A_1 v_1 = A_2 v_2$$

or

$$\left(1.0 \text{cm}^2\right) \left(2 \text{cm/s}\right) = \left(20 \text{mm}^2\right) v_2$$

or

$$v_2 = \frac{1.0 \text{cm}^2}{20 \text{mm}^2} \times 2 \text{cm/s}$$

or

$$= \frac{100 \text{mm}^2}{20 \text{mm}^2} \times 2 \text{cm/s} = 10 \text{cm/s}$$
5.5 Bernoulli’s Equation

Bernoulli’s equation relates the speed of a fluid at a point, the pressure at that point and the height of that point above a reference level. It is just the application of work-energy theorem in the case of fluid flow.

We shall consider the case of irrotational and steady flow of an incompressible and non-viscous liquid. Figure (12) shows such a flow of a liquid in a tube of varying cross-section and varying height. Consider the liquid contained between the cross-sections $A$ and $B$ of the tube. The heights of $A$ and $B$ are $h_1$ and $h_2$ respectively from a reference level. This liquid advances into the tube and after a time $\Delta t$ is contained between the cross-sections $A'$ and $B'$ as shown in figure.

![Figure 12](image)

Suppose the area of cross-section at $A = A_1$, the area of cross-section at $B = A_2$, the speed of the liquid at $A = v_1$, the speed of the liquid at $B = v_2$, the pressure at $A = P_1$, the pressure at $B = P_2$, and the density of the liquid $= \rho$.

The distance $AA' = v_1 \Delta t$ and the distance $BB' = v_2 \Delta t$. The volume between $A$ and $A'$ is $A_1 v_1 \Delta t$ and the volume between $B$ and $B'$ is $A_2 v_2 \Delta t$. By the equation of continuity,

$A_1 v_1 \Delta t = A_2 v_2 \Delta t$

The mass of this volume of liquid is

$\Delta m = \rho A_1 v_1 \Delta t = \rho A_2 v_2 \Delta t \quad \quad \quad \quad \quad \quad \cdots (i)$

Let us calculate the total work done on the part of the liquid just considered.
The forces acting on this part of the liquid are
(a) \( P_1 A_1 \) by the liquid on the left
(b) \( P_2 A_2 \) by the liquid on the right
(c) \((\Delta m)g\), weight of the liquid
(d) \( f_c \), contact forces by the walls of the tube.

In time \( \Delta t \), the point of application of \( P_1 A_1 \) is displaced by \( AA' = v_1 \Delta t \) Thus, the work done by \( P_1 A_1 \) in time \( \Delta t \) is
\[
W_1 = \left( P_1 A_1 \right) (v_1 \Delta t) = P_1 \left( \frac{\Delta m}{\rho} \right)
\]
similarly, the work done by \( P_2 A_2 \) in time \( \Delta t \) is
\[
W_2 = -\left( P_2 A_2 \right) (v_2 \Delta t) = -P_2 \left( \frac{\Delta m}{\rho} \right)
\]
The work done by the weight is equal to the negative of the change in gravitational potential energy.

Change in potential energy \((P.E)\) in time \( \Delta t \) is
\[
= P.E. \text{ of } A'B'B' - P.E. \text{ of } AA'
= P.E. A'B' + P.E. \text{ of } BB' - P.E. \text{ of } AA' - P.E. \text{ of } A'B
= P.E. \text{ of } BB' - P.E. \text{ of } AA' = (\Delta m)gh_2 - (\Delta m)gh_1
\]
Thus the work done by the weight in time \( \Delta t \) is
\[
W_3 = (\Delta m)gh_1 - (\Delta m)gh_2.
\]
The contact force \( f_c \) does no work on the liquid because it is perpendicular to the velocity.

The total work done on the liquid, in the time interval \( \Delta t \), is
\[
W = W_1 + W_2 + W_3
= P_1 \left( \frac{\Delta m}{\rho} \right) - P_2 \left( \frac{\Delta m}{\rho} \right) + (\Delta m)gh_1 - (\Delta m)gh_2 \quad \ldots \text{(ii)}
\]
The change in kinetic energy \((K.E.)\) of the same liquid in time \( \Delta t \) is
\[
= K.E. \text{ of } A'B'B' - K.E. \text{ of } AA'
= K.E. \text{ of } A'B' + K.E. \text{ of } BB' - K.E. \text{ of } AA' - K.E. \text{ of } A'B
= K.E. \text{ of } BB' - K.E. \text{ of } AA' = \frac{1}{2} (\Delta m)v_2^2 - \frac{1}{2} (\Delta m)v_1^2 \quad \ldots \text{(iii)}
\]
Since the flow is assumed to be steady, the speed at any point remains constant in time and hence the K.E. of the part A′B is same at initial and final time and cancels out when change in kinetic energy of the system is considered.

By the work-energy theorem, the total work done on the system is equal to the change in its kinetic energy. Thus,

\[ P_1 \left( \frac{\Delta m}{\rho} \right) - P_2 \left( \frac{\Delta m}{\rho} \right) + (\Delta m)gh_1 - (\Delta m)gh_2 = \frac{1}{2} (\Delta m)v_2^2 - \frac{1}{2} (\Delta m)v_1^2 \]

or

\[ \frac{P_1}{\rho} + gh_1 + \frac{1}{2} v_1^2 = \frac{P_2}{\rho} + gh_2 + \frac{1}{2} v_2^2 \]

\[ \Rightarrow P_1 + \rho gh_1 + \frac{1}{2} \rho v_1^2 = P_2 + \rho gh_2 + \frac{1}{2} \rho v_2^2 \]  

...(10)

This is known as Bernoulli’s equation.

**Example:** Figure below shows a liquid of density 1200 kg/m³ flowing steadily in a tube of varying cross-section. The cross-section at a point A is 1.0 cm² and that at B is 20 mm², the points A and B are in the same horizontal plane. The speed of the liquid at A is 10 cm/s Calculate the difference in pressures at A and B.

**Solution:** From equation of continuity, the speed v₂ at B is given by,

\[ A_1 v_1 = A_2 v_2 \]

\[ (1.0 \text{ cm}^2)(10 \text{ cm/s}) = (20 \text{ mm}^2)v_2 \]

\[ v_2 = \frac{1.0 \text{ cm}^2}{20 \text{ mm}^2} \times 10 \text{ cm/s} = 50 \text{ cm/s} \]

By Bernoulli’s equation,

\[ P_1 + \rho gh_1 + \frac{1}{2} \rho v_1^2 = P_2 + \rho gh_2 + \frac{1}{2} \rho v_2^2. \]

\[ P_1 - P_2 = \frac{1}{2} \rho v_2^2 - \frac{1}{2} \rho v_1^2 = \frac{1}{2} \times (1200 \text{ kg/m}^2) (2500 \text{ cm}^2/\text{s}^2 - 100 \text{ cm}^2/\text{s}^2) \]

\[ = 600 \text{ kg/m}^2 \times 2400 \text{ cm}^2/\text{s}^2 = 144 \text{ Pa.} \]
Applications of Bernoulli’s Equation

(a) Hydrostatics

If the speed of the fluid is zero everywhere, we get the situation of hydrostatics. Putting \( v_1 = v_2 = 0 \) in the Bernoulli’s equation (10),

\[
P_1 + \rho g h_1 = P_2 + \rho g h_2 \quad \text{or} \quad P_1 - P_2 = \rho g (h_2 - h_1)
\]
as expected from hydrostatics.

(b) Speed of Efflux

Consider a liquid of density \( \rho \) filled in a tank of large cross-sectional area \( A_1 \). There is a hole of cross-sectional area \( A_2 \) at the bottom and the liquid flows out of the tank through the hole. The situation is shown in figure (13). Suppose \( A_2 \ll A_1 \).

Let \( v_1 \) and \( v_2 \) be the speeds of the liquid at \( A_1 \) and \( A_2 \). As both the cross-sections are open to the atmosphere, the pressure there equals the atmospheric pressure \( P_0 \). If the height of the free surface above the hole is \( h \), Bernoulli’s equation gives

\[
P_0 + \frac{1}{2} \rho v_1^2 + \rho gh = P_0 + \frac{1}{2} \rho v_2^2 \quad \text{..... (i)}
\]

By the equation of continuity \( A_1 v_1 = A_2 v_2 \)

Putting \( v_1 \) in terms of \( v_2 \) in (i),

\[
\frac{1}{2} \rho \left( \frac{A_2}{A_1} \right)^2 v_1^2 + \rho gh = \frac{1}{2} \rho v_2^2 \Rightarrow \left[ 1 - \left( \frac{A_2}{A_1} \right)^2 \right] v_1^2 = 2gh
\]

If \( A_2 \ll A_1 \), this equation reduces to

\[
v_2^2 = 2gh
\]
or

\[
v_2 = \sqrt{2gh}
\]

The speed of liquid coming out through a hole at a depth \( h \) below the free surface is the same as that of a particle fallen freely through the height \( h \) under gravity. This is known as Torricelli’s theorem. The speed of the liquid coming out is called the speed of efflux.
Example: A water tank is constructed on the top of a building. With what speed will the water come out of a tap 6.0 m below the water level in the tank? Assume steady flow and that the pressure above the water level is equal to the atmospheric pressure.

Solution: The velocity is given by Torricelli’s theorem

\[ v = \sqrt{2gh} \Rightarrow v = \sqrt{2 \times (9.8 \text{ m/s}^2) \times (6.0 \text{ m})} = 11 \text{ m/s} \]

Venturi Tube

A venturi tube is used to measure the flow speed of a fluid in a tube. It consists of a constriction or a throat in the tube. As the fluid passes through the constriction, its speed increases in accordance with the equation of continuity. The pressure thus decreases as required by Bernoulli’s equation.

\[ P_1 + \frac{1}{2} \rho v_1^2 = P_2 + \frac{1}{2} \rho v_2^2 \Rightarrow (P_1 - P_2) = \frac{1}{2} \rho (v_2^2 - v_1^2) \]  

Figure (14) also shows two vertical tubes connected to the venturi tube at \( A_1 \) and \( A_2 \). If the difference in heights of the liquid levels in these tubes is \( h \), we have

\[ P_1 - P_2 = \rho g h \Rightarrow 2gh = v_2^2 - v_1^2 \]  

Knowing \( A_1 \) and \( A_2 \) one can solve equations (i) and (iii) so as to get \( v_1 \) and \( v_2 \). This allows one to know the rate of flow of liquid past a cross-section.
(d) Aspirator Pump

When a fluid passes through a region at a large speed, the pressure there decreases. This fact finds a number of useful applications. In an aspirator pump a barrel $A$ terminates in a small constriction $B$ (figure 15). The constriction is connected to a vessel containing the liquid to be sprayed through a narrow tube $C$. The air in the barrel $A$ is pushed by the operator through a piston. As the air passes through the constriction $B$, its speed is considerably increased and consequently the pressure drops. Due to reduced pressure in the constriction $B$, the liquid is raised from the vessel and is sprayed with the expelled air.

![Figure 15](image15.png)

(e) Change of Plane of Motion of a Spinning Ball

Quite often when swing bowlers of cricket deliver the ball, the ball changes its plane of motion in air. Such a deflection from the plane of projection may be explained on the basis of Bernoulli’s equation.

Suppose a ball spinning about the vertical direction is going ahead with some velocity in the horizontal direction in otherwise still air. Let us work in a frame in which the centre of the ball is at rest. In this frame the air moves past the ball at a speed $v$ in the opposite direction. The situation is shown (figure 16).

The plane of the figure represents horizontal plane. The air that goes from the $A$ side of the ball in the figure is dragged by the spin of the ball and its speed increases. The air that goes from the $B$ side of the ball in the figure suffers an opposite drag and its speed decreases. The pressure of air is reduced on the $A$ side and is increased on the $B$ side as required by the Bernoulli’s theorem. As a result, a net force $F$ acts on the ball from the $B$ side to the $A$ side due to this pressure difference. This force causes the deviation of the plane of motion.

![Figure 16](image16.png)
Example: A beaker of circular cross-section of radius 4 cm is filled with mercury up to a height of 10 cm. Find the force exerted by the mercury on the bottom of the beaker. The atmospheric pressure = $10^5$ N/m$^2$, Density of mercury = $13600$ kg/m$^3$. Take $g = 10$ m/s$^2$.

Solution: The pressure at the surface = atmospheric pressure = $10^5$ N/m$^2$

The pressure at the bottom = $10^5$ N/m$^2$ + $h \rho g = 10^5$ N/m$^2$ + \left(0.1m\right)\left(13600\,\text{kg/m}^3\right)\left(10\,\text{m/s}^2\right)$

\[= 10^5 \text{N/m}^2 + 13600 \times 10^5 \text{N/m}^2 = 1.136 \times 10^5 \text{N/m}^2\]

Force exerted by the mercury on the bottom

\[= (1.136 \times 10^5 \text{N/m}^2) \times (3.14 \times 0.04 \times 0.04 \text{m}) = 571 \text{N}\]

Example: The density of air near earth’s surface is $1.3 \text{kg/m}^2$ and the atmospheric pressure is $1.0 \times 10^5 \text{N/m}^2$. If the atmosphere had uniform density, same as that observed at the surface of the earth, what would be the height of the atmosphere to exert the same pressure?

Solution: The pressure at the surface of the earth would be

\[p = \rho gh \Rightarrow h = \frac{1.0 \times 10^5 \text{N/m}^2}{(1.3 \text{kg/m}^3)(9.8 \text{m/s}^2)} = 7850 \text{m}\]

Even Mount Everest (8848 m) would have been outside the atmosphere.

Example: A cylindrical vessel containing a liquid is closed by a smooth piston of mass $m$ as shown in the figure. The area of cross-section of the piston is $A$. If the atmospheric pressure is $P_o$, find the pressure of the liquid just below the piston.

Solution: Let the pressure of the liquid just below the piston be $P$. The forces acting on the piston are

(a) its weight, $mg$ (downward) (b) force due to the air above it, $P_oA$

(downward)

(c) force due to the liquid below it, $PA$ (upward).

If the piston is in equilibrium, $PA = P_oA + mg \Rightarrow P = P_o + \frac{mg}{A}$.
Example: A copper piece of mass 10g is suspended by a vertical spring. The spring elongates 1cm over its natural length to keep the piece in equilibrium. A beaker containing water is now placed below the piece so as to immerse the piece completely in water. Find the elongation of the spring. Density of copper 9000 kg/m³. Take g = 10 m/s².

Solution: Let the spring constant be k. When the piece is hanging in air, the equilibrium condition gives
\[ k(l\text{cm}) = (0.01\text{kg})(10\text{m/s}^2) \]
or
\[ k(l\text{cm}) = 0.1\text{N.} \quad \ldots(i) \]
The volume of the copper piece
\[ = \frac{0.01\text{kg}}{9000\text{kg/m}^3} = \frac{1}{9} \times 10^{-5} \text{m}^3 \]
This is also the volume of water displaced when the piece is immersed in water. The force of buoyancy
\[ = \text{weight of the liquid displaced} \]
\[ = \frac{1}{9} \times 10^{-5} \text{m}^3 \times (1000\text{kg/m}^3) \times (10\text{m/s}^2) \]
\[ = 0.011\text{N} \]
If the elongation of the spring is x when the piece is immersed in water, the equilibrium condition of the piece gives,
\[ kx = 0.1\text{N} - 0.011\text{N} = 0.089\text{N} \quad \ldots(ii) \]
By (i) and (ii)
\[ x = \frac{0.089}{0.1} \text{cm} = 0.89\text{cm.} \]
MCQ (Multiple Choice Questions)

Q1. A U-tube of uniform cross-sectional area, open to the atmosphere, is partially filled with mercury. Water is then poured into both arms. If the equilibrium configuration of the tube is as shown in figure, with $h_2 = 1.00 \text{ cm}$, The value of $h_1$ is
(a) 130 cm  
(b) 126 cm  
(c) 122 cm  
(d) 118 cm

Q2. A cube of wood having an edge dimension of $20.0 \text{ cm}$ and a density of $650 \text{ kg/m}^3$ floats on water. What is the distance from the horizontal top surface of the cube to the water level?
(a) 6 cm  
(b) 7 cm  
(c) 8 cm  
(d) 9 cm

Q3. A cube of wood having an edge dimension of $20.0 \text{ cm}$ and a density of $650 \text{ kg/m}^3$ floats on water. How much lead weight has to be placed on top of the cube so that its top is just level with the water?
(a) 1.8 kg  
(b) 1.2 kg  
(c) 2.4 kg  
(d) 2.8 kg

Q4. Mercury is poured into a U-tube as in figure. The left arm of the tube has cross-sectional area $A_1$ of $10.0 \text{ cm}^2$ and the right arm has a cross-sectional area $A_2$ of $5.00 \text{ cm}^2$. One hundred grams of water are then poured into the right arm as in figure. Given that the density of mercury is $13.6 \text{ g/cm}^3$, what distance $h$ does the mercury rise in the left arm?
(a) 0.49 cm  
(b) 0.32 cm  
(c) 0.63 cm  
(d) 0.41 cm
Q5. A cylindrical object of diameter 10 cm, height 20 cm and density 8000 kg/m³ is supported by a vertical spring and is half dipped in water. The elongation of the spring in equilibrium condition is
(a) 21 cm      (b) 22 cm      (c) 23 cm      (d) 24 cm

Q6. The height of a mercury barometer is 75 cm at sea level and 50 cm at the top of a hill. Ratio of density of mercury to that air is 104. The height of the hill is
(a) 250 m      (b) 2.5 km      (c) 1.25 km      (d) 750 m

Q7. A cubical block is floating in a liquid with half of its volume immersed in the liquid, where the whole system accelerates upward with acceleration of \( \frac{g}{3} \), the fraction of volume immersed in the liquid will be
(a) \( \frac{1}{2} \)      (b) \( \frac{3}{8} \)      (c) \( \frac{2}{3} \)      (d) \( \frac{3}{4} \)

Q8. A vessel contains oil (density = 0.8 g/cm³) mercury (density 13.6 g/cm³) A homogeneous sphere floats with half its volume immersed in mercury and the other half in oil. The density of the material of the sphere in g/cm³ is
(a) 3.3      (b) 6.4      (c) 7.2      (d) 12.8
NAT (Numerical Answer Type)

Q9. The area of cross-section of the two arms of a hydraulic press are 1cm² and 10cm² respectively (figure). A force of 5N is applied on the water in the thinner arm. The Force should be applied on the water in the thicker arm so that the water may remain in equilibrium is _______ N

Q10. The liquids shown in figure in the two arms are mercury specific gravity =13.6 and water. If the difference of heights of the mercury columns is 2 cm, the height h of the water column is ______ cm

Q11. A cubical block of wood of edges 3 cm float in water. The lower surface of the cube just touches the free end of a vertical spring fixed at the bottom of the pot. The maximum weight that can be put on the block without wetting it is _______ N (density of wood = 800 kg/m³ and spring constant = 50 N/m)

Q12. A spherical aluminum ball of mass 1.26 kg contains an empty spherical cavity that is concentric with the ball. The ball just barely floats in water. The radius of the cavity is _______ cm

Q13. A cylinder of height 20 m is completely filled with water. The velocity of efflux of water (in m/s) through a small hole on the side wall of the cylinder near its bottom is _______.

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MSQ (Multiple Select Questions)

Q14. A liquid is flowing through a horizontal tube. The velocities of the liquid in the two sections, which have areas of cross-section \( A_1 \) and \( A_2 \), the volumes are \( V_1 \) and \( V_2 \) respectively. The difference in the levels of the two liquid in the two vertical tubes is \( h \). Which of the statements are true?

![Diagram of liquid flow through a horizontal tube with difference in levels labeled as \( h \).]

(a) The volume of the liquid flowing through the tube in unit time is \( A_1 V_1 \)
(b) \( V_2 - V_1 = \sqrt{2gh} \)
(c) \( V_2^2 - V_1^2 = 2gh \)
(d) The total energy per unit mass of the liquid is the same in both sections of the tube.

Q15. A body floats in a liquid contained in a beaker. The whole system as shown in figure falls freely under gravity. Which of the following can not be upthrust on the body due to liquid?

(a) zero
(b) equal to the weight of the liquid displaced
(c) equal to the weight of the body in air
(d) equal to the weight of the immersed portion of the body.

Q16. Water from a tap emerges vertically downwards with an initial speed of 1.0 \( ms^{-1} \). The cross-sectional area of The tap is \( 10^{-4} \) \( m^2 \). In Assume that the pressure is constant throughout the stream of water, and that the flow is steady. The stream is 0.15 \( m \) below the tap, which of the following statements

(a) velocity of water stream at 0.15 \( m \) below the tap is 4 \( m/s \)
(b) velocity of water stream at 0.15 \( m \) below the tap is 2 \( m/s \)
(c) the cross-sectional area of the stream 0.15 \( m \) below the tap is \( 5 \times 10^{-5} \) \( m^2 \)
(d) the cross-sectional area of the stream 0.15 \( m \) below the tap is \( 2 \times 10^{-5} \) \( m^2 \)

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Q17. The spring balance 4 reads 2 kg with a block in suspended from it. KbalanoeBreads5kgwhen.a beaker with liquid is put on the pan of the balance. The two balances are now so arranged that the hanging mass is inside the liquid in the beaker as shown in the figure. In this situation:
(a) the balance $A$ will read more than 2 kg
(b) the balance $B$ will read more than 5 kg
(c) the balance $A$ will read less than 2 kg and $B$ will read more than 5 kg
(d) the balance $A$ and $B$ will read 2 kg and 5 kg respectively

Q18. A tube has two area of cross-sections as shown in figure. The diameters of the tube are $8 \text{ mm}$ and $2 \text{ mm}$. The piston is moving with a constant velocity of $0.25 \text{ m/s}$, $h = 1.25 \text{ m} \left( g = 10 \text{ m/s}^2 \right)$, which of the following statements are correct?
(a) The velocity in the horizontal direction is $4 \text{ m/s}$
(b) The velocity in the horizontal direction is $2 \text{ m/s}$
(c) The range of water falling on the horizontal surface is $4 \text{ m}$
(d) The range of water falling on the horizontal surface is $2 \text{ m}$
Q19. A non-viscous liquid of constant density 1000 kg/m³ in a streamline motion along a tube of variable cross-section. The tube is kept inclined in the vertical plane as shown in figure. The area of cross-section of the tube at two points P and Q at heights of 2 m and 5 m are respectively 4×10⁻³ m² and 8×10⁻³ m². The velocity of the liquid at point P is 1 m/s. Which of the following statements are correct for work done per unit volume by the pressure and the gravity forces as fluid flow from point P to Q?

(a) Work done per unit volume by the pressure is 29.03×10³ J/m³.
(b) Work done per unit volume by the pressure is 29.03×10⁴ J/m³.
(c) Work done per unit volume by gravity is 29.4×10³ J/m³.
(d) Work done per unit volume by gravity is 29.4×10⁴ J/m³.

Q20. A large open container of negligible mass and uniform cross-sectional area A has a small hole of cross-sectional area \( \frac{A}{100} \) in its side wall near the bottom. The container is kept on a smooth horizontal floor and contains a liquid of density \( \rho \) and mass \( m_0 \). Assuming that the liquid starts flowing out horizontally through the hole at \( t = 0 \) (where \( A = 200 \text{cm}^2, \rho = 1000 \text{kg/m}^3 \) and \( m_0 = 1000 \text{kg} \)). Which of the following statements are correct?

(a) the acceleration of the container is \( 0.4 \text{m/s}^2 \)
(b) the force is exerted on the container is \( 200 \text{N} \)
(c) fluid velocity when 75% of the liquid has drained out is \( 15.8 \text{m/s} \)
(d) the acceleration of the container is \( 0.2 \text{m/s}^2 \)
Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (b)
Solution: Let $h$ be the height of the water column added to the right side of the U-tube. Then when equilibrium is reached, the situation is as shown in the sketch at right. Now consider two points, $A$ and $B$ shown in the sketch, at the level of the water-mercury interface. By Pascal’s Principle, the absolute pressure at $B$ is the same as that at $A$. But,

$$P_A = P_0 + \rho_w g h + \rho_{Hg} g h_2$$
and

$$P_B = P_0 + \rho_w g (h_1 + h + h_2)$$

Thus, from $P_A = P_B$, $\rho_w h_1 + \rho_w h + \rho_{Hg} h_2 = \rho_w h + \rho_{Hg} h_2$ or

$$h_1 = \left[\frac{\rho_{Hg}}{\rho_w} - 1\right] h_2 = (13.6 - 1)(1.00 \text{ cm}) = 126 \text{ cm}$$

Ans. 2: (b)
Solution: 3 (a) According to Archimedes,

$$H = \rho_{\text{water}} V_{\text{water}} g = (1.00 \text{ g/cm}^3) [20.0 \times 20.0 \times (20.0 - h)] g$$

But $B = \text{Weight of block} = mg = \rho_{\text{wood}} V_{\text{wood}} g = (0.650 \text{ g/cm}^3)(20.0 \text{ cm})^2 g$

$$0.650(20.0)^3 g = 1.00(20.0)(20.0)(20.0 - h) g$$

$$20.0 - h = 20.0(0.650) \text{ so } h = 20.0(1 - 0.650) = 7.00 \text{ cm}$$

Ans. 3: (d)
Solution: $B = F_g + Mg$ where $M = \text{mass of lead}$

$$1.00(20.0)^3 g = 0.650(20.0)^3 g + Mg$$

$$M = (1.00 - 0.650)(20.0)^3 = 0.350(20.0)^3 = 2800 g = 2.80 \text{ kg}$$
Ans. 4: (a)

Solution: Using the definition of density, we have

$$h_w = \frac{m_{\text{water}}}{A_2 \rho_{\text{water}}} = \frac{100 \text{ g}}{5.00 \text{ cm}^2 \left(1.00 \text{ g/cm}^3\right)} = 20.0 \text{ cm}$$

Sketch (b) at the right represents the situation after the water is added. A volume \((A_2 h_2)\) of mercury has been displaced by water in the right tube. The additional volume mercury now in the left tube is \(A_1 h\). Since the total volume of mercury has not changed,

\[ A_2 h_2 = A_1 h \quad \text{or} \quad h_2 = \frac{A_1}{A_2} h \quad (i) \]

At the level of the mercury-water interface in the right tube, we may write the absolute pressure as:

\[ P = P_0 + \rho_{\text{water}} gh_w \]

The pressure at this same level in the left tube is given by

\[ P = P_0 + \rho_{\text{Hg}} g \left(h + h_2\right) = P_0 + \rho_{\text{water}} gh_w \]

which, using equation (i) above, reduces to

\[ \rho_{\text{Hg}} h \left[1 + \frac{A_1}{A_2}\right] = \rho_{\text{water}} h_w \quad \text{or} \quad h = \frac{\rho_{\text{water}} h_w}{\rho_{\text{Hg}} \left[1 + \frac{A_1}{A_2}\right]} \]

Thus, the level of mercury has risen a distance of

\[ h = \left(1.00 \text{ g/cm}^3\right)\left(20.0 \text{ cm}\right) \left(1+\frac{10.0}{5.00}\right) \]

\[ = 0.490 \text{ cm} \] above the original level.
Ans. 5: (d)

Solution: Volume of the cylinder 
\[ V = \pi r^2 \times H \]

\[ V = 3.14 \times 5^2 \times 20 = 1571 \text{ cm}^3 \]

Mass of the cylindrical object \((M) = \rho V\)

\[ M = \left(8000 \text{ kg/m}^3\right)\left(1571 \times 10^{-6} \text{ m}^3\right) = 12.57 \text{ kg} \]

When object is half dipped, then the volume displaced in water is \(V_d = \) Half volume of cylinder \( = 785.4 \text{ cm}^3 = 785.4 \times 10^{-6} \text{ m}^3 \)

In equilibrium conditions; \(W = F_r + T \Rightarrow F_r = W - T\)

Weight of object \(W = \rho V g\)

\[ W = \left(8000 \text{ kg/m}^3\right)\left(1571 \times 10^{-6} \text{ m}^3\right)\left(10 \text{ m/s}^2\right) = 125.7 \text{ N} \]

Force of buoyancy, \(T = \rho_a V_d g = \left(1000 \text{ kg/m}^3\right)\left(785.4 \times 10^{-6} \text{ m}^3\right)\left(10 \text{ m/s}^2\right)\)

\[ T = 7.85 \text{ N} \]

Then \(F_r = W - T = 125.7 - 7.85 = 117.85 \text{ N}\)

Also, \(F_r = Kx \Rightarrow x = \frac{F_r}{K} = \frac{117.85 \text{ N}}{500 \text{ N/m}} = 0.24 \text{ m}\)

\[ x = 24 \text{ cm} \]

Ans. 6: (b)

Solution: Difference of pressure between sea level and the top of the hill

\[ \Delta P = (h_1 - h_2) \times \rho_{tg} \times g = (75 - 50) \times 10^{-2} \times \rho_{tg} \times g \]

and pressure difference due to \(h\) meter of air \(\Delta P = h \times \rho_{air} \times g\)

By equating these two \(h \times \rho_{air} \times g = (75 - 50) \times 10^{-2} \times \rho_{tg} \times g\)

\[ h = \frac{10^{-2} \times \rho_{tg}}{\rho_{air}} = 25 \times 10^{-2} \times 10^4 = 2500 \text{ m} \]

\[ \therefore \text{ Height of the hill} = 2.5 \text{ km} \]
Ans. 7: (a)

Solution: Fraction of volume immersed in the liquid is

\[ V_{in} = \left( \frac{\rho}{\sigma} \right) V \]

where \( \rho \) is the density of block and \( \sigma \) is the density of liquid

i.e. it depends upon the densities of the block and liquid. So there will be no change in it.

If system moves upward or downward with constant velocity or acceleration.

Ans. 8: (c)

Solution: weight of floating sphere = Upthrust due to \(( Hg + \text{oil} )\)

\[ \therefore \ V \rho g = \frac{V}{2} \rho \mu g + \frac{V}{2} \rho_{\text{oil}} \times g \]

or \( \rho = \frac{\rho_{\mu} + \rho_{\text{oil}}}{2} = \frac{13.6 + 0.8}{2} = 7.2 \text{ g cm}^{-3} \)

\[ \therefore \ \text{Density of material of sphere} = 7.2 \left( \text{g cm}^{-3} \right) \]

NAT (Numerical Answer Type)

Ans. 9: 50

Solution: In equilibrium, the pressures at the two surfaces should be equal as they lie in the same horizontal level. If the atmospheric pressure is \( P \) and a force \( F \) is applied to maintain the equilibrium, the pressures are

\[ P_0 + \frac{5N}{1cm^2} \text{ and } P_0 + \frac{F}{10cm^2} \text{ respectively. This gives } F = 50N. \]

Ans. 10: 27

Solution: Suppose the atmospheric pressure = \( P_0 \)

Pressure at \( B = P_0 + \left( 0.02m \right) \left( 13600 \frac{kg}{m^3} \right) g \)

These pressures are equal as \( A \) and \( B \) are at the same horizontal level. Thus,

\[ h = \left( 0.02m \right) 3.6 = 0.27m = 27cm \]
Ans. 11: 0.36

Solution: Suppose the maximum weight that can be put without wetting it is $W$. The block in this case is completely immersed in the water. The volume of the displaced water is $(V_d)$

$$V_d = \text{Volume of block} = a^3 = (3\, cm)^3$$

$$V_d = 27\times10^{-6}\, m^3$$

The force of buoyancy $T = (\text{mass of displaced liquid})\, g$

$$T = V_d\rho_w g = (27\times10^{-6}\, m^3)(1000\, kg/m^3)(9.8\, m/s^2)$$

$$T = 0.27\, N$$

Let us calculate the portion of wood dipped in water

Height inside water $= 3\, cm \times \frac{\rho_{\text{wood}}}{\rho_{\text{water}}} = 3\, cm \times \frac{800}{1000} = 2.4\, cm$

∴ Height outside water $= 3\, cm - 2.4\, cm = 0.6\, cm$

The spring is compressed by 0.6 cm and hence the upward force exerted by the spring is

$$F_r = Kx = (50\, N/m)(0.6\times10^{-2}\, m) = 0.3\, N$$

The force of buoyancy and the spring force taken together balance the weight of the block plus the weight $W$ put on the block.

The weight of block $W' = \rho_{\text{wood}} \times V \times g$

$$W' = (800\, kg/m^3)(27\times10^{-6}\, m^3)(9.8\, m/s^2)$$

$$W' = 0.21\, N$$

Thus upward force = downward force

$$T + F_r = W + W'$$

$$0.27 + 0.3 = W + 0.21 \Rightarrow W = 0.36\, N$$
Ans. 12: 5.74

Solution: The weight of the ball must be equal to the buoyant force of the water:

\[ 1.26 \text{kg} = \rho_{\text{water}} \frac{4}{3} \pi r_{\text{outer}}^3 g \]

\[ r_{\text{outer}} = \left( \frac{3 \times 1.26 \text{kg}}{4 \pi 1000 \text{ kg/m}^3} \right)^{1/3} = 6.70 \text{ cm} \]

The mass of the ball is determined by the density of aluminum:

\[ m = \rho_A V = \rho_A \left( \frac{4}{3} \pi r_0^3 - \frac{4}{3} \pi r_i^3 \right) \]

\[ 1.26 \text{kg} = 2700 \text{ kg/m}^3 \left( \frac{4}{3} \pi \right) \left( 0.067 \text{ m}^3 - r_i^3 \right) \]

\[ 1.11 \times 10^{-4} \text{m}^3 = 3.01 \times 10^{-4} \text{m}^3 - r_i^3 \]

\[ r_i = \left( 1.89 \times 10^{-4} \text{m}^3 \right)^{1/3} = 5.74 \text{ cm} \]

Ans. 13: 20

Solution: The velocity of efflux is

\[ V = \sqrt{2gh} = \sqrt{2 \times 10 \times 10} = 20 \text{ m/s} \]

MSQ (Multiple Select Questions)

Ans. 14: (a), (c) and (d)

Solution: By the equation of continuity \( A_1 V_1 = A_2 V_2 \)

Thus the volume of the liquid flowing through the tube in unit time is same and equal to \( A_1 V_1 \).

According to Bernoulli’s equation which also the consequence of conservation of energy states

\[ P_1 + \frac{1}{2} \rho V_1^2 = P_2 + \frac{1}{2} \rho V_2^2 \]

and \( P_1 - P_2 = \rho gh \)

Thus \( V_2^2 - V_1^2 = 2gh \)

Thus correct statements are (a), (c) and (d).
Ans. 15: (b), (c) and (d)

Solution: Upthrust = \( V \rho_{\text{liquid}} \times (g - a) \)

Where \( a = \) downward acceleration
\( g = \) acceleration due to gravity
\( V = \) volume of the liquid displaced
\( \rho_{\text{liquid}} = \) density of the liquid

In case of free fall, \( a = g \)

Thus, upthrust = \( V \rho_{\text{liquid}} (g - g) = 0 \)

Ans. 16: (b) and (c)

Solution: By equation of continuity \( v_1 A_1 = v_2 A_2 \)

\( v_2 = \) velocity of water stream at 0.15 m below the tap

now \( v_2^2 = (1)^2 + 2\times10\times0.15 = 4 \) or \( v_2 = 2 \text{ m/s} \)

\[ A_2 = \frac{v_1 A_1}{v_2} = \frac{1\times10^{-4}}{2} = 5\times10^{-5} \text{ m}^2 \]

\[ \therefore \text{cross-section area of stream} = 5\times10^{-5} \text{ m}^2 \]

Ans. 17: (b) and (c)

Solution: When the hanging mass is inside the liquid, the liquid will apply an upthrust on the mass. Hence the balance \( A \) will read less than 2 kg. Infact \( m \) loses a weight when dipped in a liquid.

An upthrust acts on block \( m \)

By reaction, an equal force will be exerted on the liquid contained in beaker in the downward direction.

Hence \( B \) will read more than 5 kg.

Hence (b) and (c) both are correct

Infact (b) is contained in (c). So it is adequate to say that (c) holds good.
Ans. 18: (a) and (d)

Solution: By equation of continuity,

\[ A_1 v_1 = A_2 v_2 \]

\[ A_1 = \pi \times (4 \times 10^{-3})^2 \text{ m}^2 = \pi \times 16 \times 10^{-6} \text{ m}^2 \]

\[ A_2 = \pi \times (1 \times 10^{-3})^2 \text{ m}^2 = \pi \times 10^{-6} \text{ m}^2 \]

\[ v_1 = 0.25 \text{ m/s} \]

\[ \therefore v_2 = \frac{A_1 v_1}{A_2} \quad \text{or} \quad v_2 = \frac{\left(\pi \times 16 \times 10^{-6}\right) \times (0.25)}{\left(\pi \times 10^{-6}\right)} \]

or \[ v_2 = 16 \times 0.25 = 4 \text{ m/s} \]

this velocity \( v_2 \) is in horizontal direction

vertical height = \( h \), time = \( t \), Path of water = Parabola

\[ \therefore \frac{1}{2} g t^2 = h \quad \text{or} \quad t = \sqrt{\frac{2h}{g}} \] (ii)

\[ \therefore \text{Horizontal distance or range} = v_2 t \]

\[ \therefore \text{Range} = 4 \times \sqrt{\frac{2 \times 1.25}{10}} \quad \text{or} \quad \text{Range} = \frac{4 \times 1}{2} \]

or \[ \text{Range} = 2 \text{ m} \]

Ans. 19: (a) and (e)

Solution: According to equation of continuity,

\[ A_1 v_1 = A_2 v_2 \quad \text{or} \quad v_2 = \frac{A_1 v_1}{A_2} = \frac{\left(4 \times 10^{-2}\right) \times (1)}{\left(8 \times 10^{-2}\right)} = \frac{1}{2} \]

\[ \therefore v_2 = \frac{1}{2} \text{ m/s} \] (i)

According to Bernoulli’s theorem,

\[ P_1 + \frac{1}{2} \rho v_1^2 + \rho gh_1 = P_2 + \frac{1}{2} \rho v_2^2 + \rho gh_2 \]
or \( (P_1 - P_2) = \rho g (h_2 - h_1) + \frac{1}{2} \rho (v_2^2 - v_1^2) \) \hspace{1cm} (ii)

(i) Work done per unit volume by the pressure:

\[ W_1 = P_1 - P_2 \]

From equation (ii), we get

\[ W_1 = \rho g (h_2 - h_1) + \frac{1}{2} \rho (v_2^2 - v_1^2). \] Put \( v_2 \) from (i)

\[ \therefore W_1 = (10^3)(9.8)(5-2) + \frac{1}{2}(10^3)(\frac{1}{4}-1) \]

\[ \text{or} \quad W_1 = (29400 - 375) = 29.025 \times 10^3 \text{ J/m}^3 \]

\[ \text{or} \quad W_1 = 29.025 \times 10^3 \text{ J/m}^3 \]

\[ \therefore \text{Work done per unit volume by pressure} = 29.025 \times 10^3 \text{ J/m}^3 \]

(ii) Work done per unit volume by gravity:

\[ W_2 = \rho g (h_2 - h_1) \]

\[ \text{or} \quad W_2 = (10^3)(9.8)(5-2) = 29400 \text{ J/m}^3 \]

\[ \text{or} \quad W_2 = 29.4 \times 10^3 \text{ J/m}^3 \]

Work done per unit volume by gravity = 29.4 \times 10^3 \text{ J/m}^3

Ans. 20: (b), (c) and (d)

Solution: (i) Acceleration of container:

Mass of water = volume \times density

\[ \therefore m_0 = (AH) \rho \]

or

\[ H = \frac{m_0}{A \rho} \]

\[ \therefore \text{velocity of efflux} = v \]

\[ \therefore v = \sqrt{gh} = \sqrt{2g \times \frac{m_0}{A \rho}} \]
When the liquid flows out of the container horizontally, a force is exerted on the container.

Force \( F = \rho \times (\text{area of hole}) \times v^2 \)

or \( F = \rho \times \frac{A}{100} \times v^2 \)

or \( F = \frac{\rho A}{100} \times \frac{2m_0 g}{A \rho} \), from (ii)

or \( m_0 a = \frac{m_0 g}{50} \), \( \therefore F = a_0 a \) or \( a = \frac{g}{50} \)

(ii) \( F = m_0 a = 1000 \times 0.2 = 200 \, N \)

(iii) To calculate \( v \) when 75% of liquid has drained out:

in this position, \( h = \frac{H}{4} \)

\( \therefore v = \sqrt{2 gh} \) or \( v = \sqrt{2g \frac{H}{4}} \)

or \( v = \sqrt{\frac{g \times m_0}{2 A \rho}} \) from (i)

or \( v = \sqrt{\frac{m_0 g}{2 A \rho}} = \sqrt{\frac{1000 \times 10}{2 \times 2 \times 10^{-2} \times 1000}} = \sqrt{250} = 15.8 \, m/s \)